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Several Properties of the σ-additive Measure

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Summary. A continuation of [5]. The paper contains the definition and basic properties of a σ-additive, nonnegative measure, with values in $\overline{\mathbb{R}}$, the enlarged set of real numbers, where $\overline{\mathbb{R}}$ denotes set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ - by R.Sikorski [12]. Some simple theorems concerning basic properties of a σ-additive measure, measurable sets, measure zero sets are proved. The work is the fourth part of the series of articles concerning the Lebesgue measure theory.

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The terminology and notation used here have been introduced in the following papers: [14], [13], [8], [9], [6], [7], [1], [11], [2], [10], [3], [4], and [5]. The following proposition is true

(1) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every function F from $\mathbb{N}$ into S holds $M \cdot F$ is non-negative.

The scheme *RecExFun* concerns a set $\mathcal{A}$, a σ-field $\mathcal{B}$ of subsets of $\mathcal{A}$, an element $\mathcal{C}$ of $\mathcal{B}$, and a ternary predicate $\mathcal{P}$, and states that:

there exists a function f from $\mathbb{N}$ into $\mathcal{B}$ such that $f(0) = \mathcal{C}$ and for every element n of $\mathbb{N}$ holds $\mathcal{P}[n, f(n), f(n+1)]$

provided the following conditions are satisfied:

- for every natural number n and for every element x of $\mathcal{B}$ there exists an element y of $\mathcal{B}$ such that $\mathcal{P}[n, x, y]$,
- for every natural number n and for all elements x, y_1, y_2 of $\mathcal{B}$ such that $\mathcal{P}[n, x, y_1]$ and $\mathcal{P}[n, x, y_2]$ holds $y_1 = y_2$.

Let X be a set, and let S be a σ-field of subsets of X. A denumerable family of subsets of X is called a family of measureable sets of S if:

(Def.1) it $\subseteq S$.

One can prove the following propositions:

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(2) For every set X and for every σ-field S of subsets of X and for every denumerable family T of subsets of X holds T is a family of measureable sets of S if and only if $T \subseteq S$.

(3) For every set X and for every σ-field S of subsets of X and for every family T of measureable sets of S holds $\bigcap T \in S$ and $\bigcup T \in S$.

Let X be a set, and let S be a σ-field of subsets of X, and let T be a family of measureable sets of S. Then $\bigcap T$ is an element of S.

Let X be a set, and let S be a σ-field of subsets of X, and let T be a family of measureable sets of S. Then $\bigcup T$ is an element of S.

Let X be a set, and let S be a σ-field of subsets of X, and let F be a function from $\mathbb{N}$ into S, and let n be an element of $\mathbb{N}$. Then $F(n)$ is an element of S.

One can prove the following propositions:

(4) For every set X and for every σ-field S of subsets of X and for every function N from $\mathbb{N}$ into S there exists a function F from $\mathbb{N}$ into S such that $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus N(n)$.

(5) For every set X and for every σ-field S of subsets of X and for every function N from $\mathbb{N}$ into S there exists a function F from $\mathbb{N}$ into S such that $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \cup F(n)$.

(6) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \cup F(n)$. Then for an arbitrary r and for every natural number n holds $r \in F(n)$ if and only if there exists a natural number k such that $k \leq n$ and $r \in N(k)$.

(7) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Then for every function F from $\mathbb{N}$ into S such that $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \cup F(n)$ for all natural numbers n, m such that $n < m$ holds $F(n) \subseteq F(m)$.

(8) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Let G be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose that

(i) $G(0) = N(0)$,
(ii) for every element n of $\mathbb{N}$ holds $G(n+1) = N(n+1) \cup G(n)$,
(iii) $F(0) = N(0)$,
(iv) for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus G(n)$.

Then for all natural numbers n, m such that $n \leq m$ holds $F(n) \subseteq G(m)$.

(9) For every set X and for every σ-field S of subsets of X and for every function N from $\mathbb{N}$ into S and for every function G from $\mathbb{N}$ into S there exists a function F from $\mathbb{N}$ into S such that $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus G(n)$.

(10) For every set X and for every σ-field S of subsets of X and for every function N from $\mathbb{N}$ into S there exists a function F from $\mathbb{N}$ into S such

that $F(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(0) \setminus N(n)$.

(11) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Let G be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose that

(i) $G(0) = N(0)$,

(ii) for every element n of $\mathbb{N}$ holds $G(n+1) = N(n+1) \cup G(n)$,

(iii) $F(0) = N(0)$,

(iv) for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus G(n)$.

Then for all natural numbers n, m such that $n < m$ holds $F(n) \cap F(m) = \emptyset$.

(12) For every set X and for every σ-field S of subsets of X and for every function N from $\mathbb{N}$ into S and for every element n of $\mathbb{N}$ holds $N(n) \in \operatorname{rng} N$.

(13) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every family T of measureable sets of S and for every function F from $\mathbb{N}$ into S such that $T = \operatorname{rng} F$ holds $M(\bigcup T) \leq \sum(M \cdot F)$.

(14) For every set X and for every σ-field S of subsets of X and for every family T of measureable sets of S there exists a function F from $\mathbb{N}$ into S such that $T = \operatorname{rng} F$.

(15) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Then if $F(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(0) \setminus N(n)$ and $N(n+1) \subseteq N(n)$, then for every element n of $\mathbb{N}$ holds $F(n) \subseteq F(n+1)$.

(16) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every family T of measureable sets of S such that for every set A such that $A \in T$ holds A is a set of measure zero w.r.t. M holds $\bigcup T$ is a set of measure zero w.r.t. M.

(17) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every family T of measureable sets of S such that there exists a set A such that $A \in T$ and A is a set of measure zero w.r.t. M holds $\bigcap T$ is a set of measure zero w.r.t. M.

(18) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every family T of measureable sets of S such that for every set A such that $A \in T$ holds A is a set of measure zero w.r.t. M holds $\bigcap T$ is a set of measure zero w.r.t. M.

Let X be a set, and let S be a σ-field of subsets of X. A family of measureable sets of S is called a family of measureable non-decrement sets of S if:

(Def.2) there exists a function F from $\mathbb{N}$ into S such that it $= \operatorname{rng} F$ and for every element n of $\mathbb{N}$ holds $F(n) \subseteq F(n+1)$.

We now state the proposition

(19) For every set X and for every σ-field S of subsets of X and for every family T of measureable sets of S holds T is a family of measureable non-

decrement sets of S if and only if there exists a function F from $\mathbb{N}$ into S such that $T = \operatorname{rng} F$ and for every element n of $\mathbb{N}$ holds $F(n) \subseteq F(n+1)$.

Let X be a set, and let S be a σ-field of subsets of X. A family of measureable sets of S is called a family of measureable non-increment sets of S if:

(Def.3) there exists a function F from $\mathbb{N}$ into S such that it $= \operatorname{rng} F$ and for every element n of $\mathbb{N}$ holds $F(n+1) \subseteq F(n)$.

We now state several propositions:

(20) For every set X and for every σ-field S of subsets of X and for every family T of measureable sets of S holds T is a family of measureable non-increment sets of S if and only if there exists a function F from $\mathbb{N}$ into S such that $T = \operatorname{rng} F$ and for every element n of $\mathbb{N}$ holds $F(n+1) \subseteq F(n)$.

(21) Let X be a set. Let S be a σ-field of subsets of X. Then for every function N from $\mathbb{N}$ into S and for every function F from $\mathbb{N}$ into S such that $F(0) = \emptyset$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(0) \setminus N(n)$ and $N(n+1) \subseteq N(n)$ holds $\operatorname{rng} F$ is a family of measureable non-decrement sets of S.

(22) For every set X and for every non-empty family S of subsets of X and for every function N from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $N(n) \subseteq N(n+1)$ for all natural numbers m, n such that $n < m$ holds $N(n) \subseteq N(m)$.

(23) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus N(n)$ and $N(n) \subseteq N(n+1)$. Then for all natural numbers n, m such that $n < m$ holds $F(n) \cap F(m) = \emptyset$.

(24) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Then for every function F from $\mathbb{N}$ into S such that $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus N(n)$ and $N(n) \subseteq N(n+1)$ holds $\bigcup \operatorname{rng} F = \bigcup \operatorname{rng} N$.

(25) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Then for every function F from $\mathbb{N}$ into S such that $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus N(n)$ and $N(n) \subseteq N(n+1)$ holds F is a sequence of separated subsets of S.

(26) Let X be a set. Let S be a σ-field of subsets of X. Let N be a function from $\mathbb{N}$ into S. Let F be a function from $\mathbb{N}$ into S. Suppose $F(0) = N(0)$ and for every element n of $\mathbb{N}$ holds $F(n+1) = N(n+1) \setminus N(n)$ and $N(n) \subseteq N(n+1)$. Then $N(0) = F(0)$ and for every element n of $\mathbb{N}$ holds $N(n+1) = F(n+1) \cup N(n)$.

(27) For every set X and for every σ-field S of subsets of X and for every σ-measure M on S and for every function F from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $F(n) \subseteq F(n+1)$ holds $M(\bigcup \operatorname{rng} F) = \sup \operatorname{rng}(M \cdot F)$.

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