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Metrics in the Cartesian Product - Part II

Stanisława Kanas
Technical University of Rzeszów

Adam Lecko
Technical University of Rzeszów

Summary. A continuation of [9]. It deals with the method of creation of the distance in the Cartesian product of metric spaces. The distance between two points belonging to Cartesian product of metric spaces has been defined as square root of the sum of squares of distances of appriopriate coordinates (or projections) of these points. It is shown that the product of metric spaces with such a distance is a metric space. Examples of metric spaces defined in this way are given.

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The articles [7], [15], [4], [5], [2], [6], [1], [10], [11], [3], [8], [13], [12], [14], and [9] provide the terminology and notation for this paper. We adopt the following convention: X, Y are metric spaces, x_1, y_1, z_1 are elements of the carrier of X, and x_2, y_2, z_2 are elements of the carrier of Y. Let us consider X, Y. The functor $\rho^{[:X,Y:]}$ yields a function from $[:[:$ the carrier of X, the carrier of $Y:]$, $[:$ the carrier of X, the carrier of $Y:]:]$ into $\mathbb{R}$ and is defined by:

(Def.1) for all elements x_1, y_1 of the carrier of X and for all elements x_2, y_2 of the carrier of Y and for all elements x, y of $[:$ the carrier of X, the carrier of $Y:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ holds $\rho^{[:X,Y:]}(x, y) = \sqrt{(\rho(x_1,y_1))^{\mathbf{2}} + (\rho(x_2,y_2))^{\mathbf{2}}}$.

Next we state the proposition

(1) Let X be a metric space. Let Y be a metric space. Let F be a function from $[:[:$ the carrier of X, the carrier of $Y:]$, $[:$ the carrier of X, the carrier of $Y:]:]$ into $\mathbb{R}$. Then $F = \rho^{[:X,Y:]}$ if and only if for all elements x_1, y_1 of the carrier of X and for all elements x_2, y_2 of the carrier of Y and for all elements x, y of $[:$ the carrier of X, the carrier of $Y:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ holds $F(x, y) = \sqrt{(\rho(x_1,y_1))^{\mathbf{2}} + (\rho(x_2,y_2))^{\mathbf{2}}}$.

Next we state several propositions:

(2) For all elements a, b of $\mathbb{R}$ such that $0 \leq a$ and $0 \leq b$ holds $\sqrt{a+b} = 0$ if and only if $a = 0$ and $b = 0$.

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(3) For all elements x, y of $[\!:$ the carrier of X, the carrier of $Y :\!]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $\rho^{[:X,Y:]}(x, y) = 0$ if and only if $x = y$.

(4) For all elements x, y of $[\!:$ the carrier of X, the carrier of $Y :\!]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $\rho^{[:X,Y:]}(x, y) = \rho^{[:X,Y:]}(y, x)$.

(5) For all elements a, b, c, d of $\mathbb{R}$ such that $0 \leq a$ and $0 \leq b$ and $0 \leq c$ and $0 \leq d$ holds $\sqrt{(a+c)^{\mathbf{2}} + (b+d)^{\mathbf{2}}} \leq \sqrt{a^{\mathbf{2}} + b^{\mathbf{2}}} + \sqrt{c^{\mathbf{2}} + d^{\mathbf{2}}}$.

(6) For all elements x, y, z of $[\!:$ the carrier of X, the carrier of $Y :\!]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ and $z = \langle z_1, z_2 \rangle$ holds $\rho^{[:X,Y:]}(x, z) \leq \rho^{[:X,Y:]}(x, y) + \rho^{[:X,Y:]}(y, z)$.

Let us consider X, Y, and let x, y be elements of $[\!:$ the carrier of X, the carrier of $Y :\!]$. The functor $\rho^{\mathbf{2}}(x,y)$ yielding a real number is defined as follows:

(Def.2) $\rho^{\mathbf{2}}(x,y) = \rho^{[:X,Y:]}(x, y)$.

Next we state the proposition

(7) For all elements x, y of $[\!:$ the carrier of X, the carrier of $Y :\!]$ holds $\rho^{\mathbf{2}}(x,y) = \rho^{[:X,Y:]}(x, y)$.

Let X, Y be metric spaces. The functor $[\!:X,Y:\!]$ yielding a metric space is defined as follows:

(Def.3) $[\!:X,Y:\!] = \langle [\!:$ the carrier of X, the carrier of $Y :\!], \rho^{[:X,Y:]}\rangle$.

We now state the proposition

(8) For every metric space X and for every metric space Y holds $\langle [\!:$ the carrier of X, the carrier of $Y :\!], \rho^{[:X,Y:]}\rangle$ is a metric space.

In the sequel Z will be a metric space and x_3, y_3, z_3 will be elements of the carrier of Z. Let us consider X, Y, Z. The functor $\rho^{[:X,Y,Z:]}$ yielding a function from $[\!:[\!:$ the carrier of X, the carrier of Y, the carrier of $Z :\!], [\!:$ the carrier of X, the carrier of Y, the carrier of $Z :\!]:\!]$ into $\mathbb{R}$ is defined by the condition (Def.4).

(Def.4) Let x_1, y_1 be elements of the carrier of X. Let x_2, y_2 be elements of the carrier of Y. Let x_3, y_3 be elements of the carrier of Z. Then for all elements x, y of $[\!:$ the carrier of X, the carrier of Y, the carrier of $Z :\!]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $\rho^{[:X,Y,Z:]}(x, y) = \sqrt{(\rho(x_1,y_1))^{\mathbf{2}} + (\rho(x_2,y_2))^{\mathbf{2}} + (\rho(x_3,y_3))^{\mathbf{2}}}$.

One can prove the following propositions:

(9) Let X be a metric space. Let Y be a metric space. Let Z be a metric space. Let F be a function from $[\!:[\!:$ the carrier of X, the carrier of Y, the carrier of $Z :\!], [\!:$ the carrier of X, the carrier of Y, the carrier of $Z :\!]:\!]$ into $\mathbb{R}$. Then $F = \rho^{[:X,Y,Z:]}$ if and only if for all elements x_1, y_1 of the carrier of X and for all elements x_2, y_2 of the carrier of Y and for all elements x_3, y_3 of the carrier of Z and for all elements x, y of $[\!:$ the carrier of X, the carrier of Y, the carrier of $Z :\!]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $F(x, y) = \sqrt{(\rho(x_1,y_1))^{\mathbf{2}} + (\rho(x_2,y_2))^{\mathbf{2}} + (\rho(x_3,y_3))^{\mathbf{2}}}$.

(10) For all elements x, y of $[:$ the carrier of X, the carrier of Y, the carrier of $Z:]$ such that $x = \langle x_1, x_2, x_3\rangle$ and $y = \langle y_1, y_2, y_3\rangle$ holds $\rho^{[:X,Y,Z:]}(x, y) = 0$ if and only if $x = y$.

(11) For all elements x, y of $[:$ the carrier of X, the carrier of Y, the carrier of $Z:]$ such that $x = \langle x_1, x_2, x_3\rangle$ and $y = \langle y_1, y_2, y_3\rangle$ holds $\rho^{[:X,Y,Z:]}(x, y) = \rho^{[:X,Y,Z:]}(y, x)$.

(12) For all elements a, b, c of $\mathbb{R}$ holds $(a + b + c)^{\mathbf{2}} = a^{\mathbf{2}} + b^{\mathbf{2}} + c^{\mathbf{2}} + (2 \cdot a \cdot b + 2 \cdot a \cdot c + 2 \cdot b \cdot c)$.

(13) Let a, b, c, d, e, f be elements of $\mathbb{R}$. Suppose $0 \leq a$ and $0 \leq b$ and $0 \leq c$ and $0 \leq d$ and $0 \leq e$ and $0 \leq f$. Then $2 \cdot (a \cdot d) \cdot (c \cdot b) + 2 \cdot (a \cdot f) \cdot (e \cdot c) + 2 \cdot (b \cdot f) \cdot (e \cdot d) \leq (a \cdot d)^{\mathbf{2}} + (c \cdot b)^{\mathbf{2}} + (a \cdot f)^{\mathbf{2}} + (e \cdot c)^{\mathbf{2}} + (b \cdot f)^{\mathbf{2}} + (e \cdot d)^{\mathbf{2}}$.

(14) Let a, b, c, d, e, f be elements of $\mathbb{R}$. Then $a^{\mathbf{2}} \cdot d^{\mathbf{2}} + (a^{\mathbf{2}} \cdot f^{\mathbf{2}} + c^{\mathbf{2}} \cdot b^{\mathbf{2}}) + e^{\mathbf{2}} \cdot c^{\mathbf{2}} + b^{\mathbf{2}} \cdot f^{\mathbf{2}} + e^{\mathbf{2}} \cdot d^{\mathbf{2}} + e^{\mathbf{2}} \cdot f^{\mathbf{2}} + b^{\mathbf{2}} \cdot d^{\mathbf{2}} + a^{\mathbf{2}} \cdot c^{\mathbf{2}} = (a^{\mathbf{2}} + b^{\mathbf{2}} + e^{\mathbf{2}}) \cdot (c^{\mathbf{2}} + d^{\mathbf{2}} + f^{\mathbf{2}})$.

(15) Let a, b, c, d, e, f be elements of $\mathbb{R}$. Suppose $0 \leq a$ and $0 \leq b$ and $0 \leq c$ and $0 \leq d$ and $0 \leq e$ and $0 \leq f$. Then $(a \cdot c + b \cdot d + e \cdot f)^{\mathbf{2}} \leq (a^{\mathbf{2}} + b^{\mathbf{2}} + e^{\mathbf{2}}) \cdot (c^{\mathbf{2}} + d^{\mathbf{2}} + f^{\mathbf{2}})$.

(16) Let x, y, z be elements of $[:$ the carrier of X, the carrier of Y, the carrier of $Z:]$. Then if $x = \langle x_1, x_2, x_3\rangle$ and $y = \langle y_1, y_2, y_3\rangle$ and $z = \langle z_1, z_2, z_3\rangle$, then $\rho^{[:X,Y,Z:]}(x, z) \leq \rho^{[:X,Y,Z:]}(x, y) + \rho^{[:X,Y,Z:]}(y, z)$.

Let us consider X, Y, Z, and let x, y be elements of $[:$ the carrier of X, the carrier of Y, the carrier of $Z:]$. The functor $\rho^{\mathbf{3}}(x,y)$ yielding a real number is defined as follows:

(Def.5) $\rho^{\mathbf{3}}(x,y) = \rho^{[:X,Y,Z:]}(x, y)$.

One can prove the following proposition

(17) For all elements x, y of $[:$ the carrier of X, the carrier of Y, the carrier of $Z:]$ holds $\rho^{\mathbf{3}}(x,y) = \rho^{[:X,Y,Z:]}(x, y)$.

Let X, Y, Z be metric spaces. The functor $[:X, Y:]$ yields a metric space and is defined by:

(Def.6) $[:X, Y:] = \langle [:$ the carrier of X, the carrier of Y, the carrier of $Z:]$, $\rho^{[:X,Y,Z:]}\rangle$.

The following proposition is true

(18) For every metric space X and for every metric space Y and for every metric space Z holds $\langle [:$ the carrier of X, the carrier of Y, the carrier of $Z:], \rho^{[:X,Y,Z:]}\rangle$ is a metric space.

In the sequel x_1, x_2, y_1, y_2, z_1, z_2 denote elements of $\mathbb{R}$. The function $\rho^{[:\mathbb{R},\mathbb{R}:]}$ from $[:[:\mathbb{R}, \mathbb{R}:], [:\mathbb{R}, \mathbb{R}:]:]$ into $\mathbb{R}$ is defined by:

(Def.7) for all elements x_1, y_1, x_2, y_2 of $\mathbb{R}$ and for all elements x, y of $[:\mathbb{R}, \mathbb{R}:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R}:]}(x, y) = \rho_{\mathbb{R}}(x_1, y_1) + \rho_{\mathbb{R}}(x_2, y_2)$.

The following propositions are true:

(19) For all elements x_1, x_2, y_1, y_2 of $\mathbb{R}$ and for all elements x, y of $[\!: \mathbb{R}, \mathbb{R} :\!]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R}:]}(x, y) = 0$ if and only if $x = y$.

(20) For all elements x, y of $[\!: \mathbb{R}, \mathbb{R} :\!]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R}:]}(x, y) = \rho^{[:\mathbb{R},\mathbb{R}:]}(y, x)$.

(21) For all elements x, y, z of $[\!: \mathbb{R}, \mathbb{R} :\!]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ and $z = \langle z_1, z_2 \rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R}:]}(x, z) \leq \rho^{[:\mathbb{R},\mathbb{R}:]}(x, y) + \rho^{[:\mathbb{R},\mathbb{R}:]}(y, z)$.

The metric space $[\!: \mathbb{R}_{\mathrm{M}}, \mathbb{R}_{\mathrm{M}} :\!]$ is defined by:

(Def.8) $[\!: \mathbb{R}_{\mathrm{M}}, \mathbb{R}_{\mathrm{M}} :\!] = \langle [\!: \mathbb{R}, \mathbb{R} :\!], \rho^{[:\mathbb{R},\mathbb{R}:]} \rangle$.

The function $\rho^{\mathbb{R}^2}$ from $[\!: [\!: \mathbb{R}, \mathbb{R} :\!], [\!: \mathbb{R}, \mathbb{R} :\!] :\!]$ into $\mathbb{R}$ is defined as follows:

(Def.9) for all elements x_1, y_1, x_2, y_2 of $\mathbb{R}$ and for all elements x, y of $[\!: \mathbb{R}, \mathbb{R} :\!]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $\rho^{\mathbb{R}^2}(x, y) = \sqrt{\rho_{\mathbb{R}}(x_1, y_1)^{\mathbf{2}} + \rho_{\mathbb{R}}(x_2, y_2)^{\mathbf{2}}}$.

We now state three propositions:

(22) For all elements x_1, x_2, y_1, y_2 of $\mathbb{R}$ and for all elements x, y of $[\!: \mathbb{R}, \mathbb{R} :\!]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $\rho^{\mathbb{R}^2}(x, y) = 0$ if and only if $x = y$.

(23) For all elements x, y of $[\!: \mathbb{R}, \mathbb{R} :\!]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $\rho^{\mathbb{R}^2}(x, y) = \rho^{\mathbb{R}^2}(y, x)$.

(24) For all elements x, y, z of $[\!: \mathbb{R}, \mathbb{R} :\!]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ and $z = \langle z_1, z_2 \rangle$ holds $\rho^{\mathbb{R}^2}(x, z) \leq \rho^{\mathbb{R}^2}(x, y) + \rho^{\mathbb{R}^2}(y, z)$.

The Euclidean plain being a metric space is defined as follows:

(Def.10) the Euclidean plain$= \langle [\!: \mathbb{R}, \mathbb{R} :\!], \rho^{\mathbb{R}^2} \rangle$.

In the sequel x_3, y_3, z_3 denote elements of $\mathbb{R}$. The function $\rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}$ from $[\!: [\!: \mathbb{R}, \mathbb{R}, \mathbb{R} :\!], [\!: \mathbb{R}, \mathbb{R}, \mathbb{R} :\!] :\!]$ into $\mathbb{R}$ is defined by the condition (Def.11).

(Def.11) Let $x_1, y_1, x_2, y_2, x_3, y_3$ be elements of $\mathbb{R}$. Then for all elements x, y of $[\!: \mathbb{R}, \mathbb{R}, \mathbb{R} :\!]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(x, y) = \rho_{\mathbb{R}}(x_1, y_1) + \rho_{\mathbb{R}}(x_2, y_2) + \rho_{\mathbb{R}}(x_3, y_3)$.

We now state three propositions:

(25) For all elements $x_1, x_2, y_1, y_2, x_3, y_3$ of $\mathbb{R}$ and for all elements x, y of $[\!: \mathbb{R}, \mathbb{R}, \mathbb{R} :\!]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(x, y) = 0$ if and only if $x = y$.

(26) For all elements x, y of $[\!: \mathbb{R}, \mathbb{R}, \mathbb{R} :\!]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(x, y) = \rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(y, x)$.

(27) For all elements x, y, z of $[\!: \mathbb{R}, \mathbb{R}, \mathbb{R} :\!]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ and $z = \langle z_1, z_2, z_3 \rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(x, z) \leq \rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(x, y) + \rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(y, z)$.

The metric space $[\!: \mathbb{R}_{\mathrm{M}}, \mathbb{R}_{\mathrm{M}}, \mathbb{R}_{\mathrm{M}} :\!]$ is defined as follows:

(Def.12) $[\!: \mathbb{R}_{\mathrm{M}}, \mathbb{R}_{\mathrm{M}}, \mathbb{R}_{\mathrm{M}} :\!] = \langle [\!: \mathbb{R}, \mathbb{R}, \mathbb{R} :\!], \rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]} \rangle$.

The function $\rho^{\mathbb{R}^3}$ from $[\![[\![\mathbb{R}, \mathbb{R}, \mathbb{R}]\!], [\![\mathbb{R}, \mathbb{R}, \mathbb{R}]\!]]\!]$ into $\mathbb{R}$ is defined by the condition (Def.13).

(Def.13) Let x_1, y_1, x_2, y_2, x_3, y_3 be elements of $\mathbb{R}$. Then for all elements x, y of $[\![\mathbb{R}, \mathbb{R}, \mathbb{R}]\!]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $\rho^{\mathbb{R}^3}(x, y) = \sqrt{\rho_{\mathbb{R}}\,(x_1, y_1)^{\mathbf{2}} + \rho_{\mathbb{R}}\,(x_2, y_2)^{\mathbf{2}} + \rho_{\mathbb{R}}\,(x_3, y_3)^{\mathbf{2}}}$.

One can prove the following three propositions:

(28) For all elements x_1, x_2, y_1, y_2, x_3, y_3 of $\mathbb{R}$ and for all elements x, y of $[\![\mathbb{R}, \mathbb{R}, \mathbb{R}]\!]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $\rho^{\mathbb{R}^3}(x, y) = 0$ if and only if $x = y$.

(29) For all elements x, y of $[\![\mathbb{R}, \mathbb{R}, \mathbb{R}]\!]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ holds $\rho^{\mathbb{R}^3}(x, y) = \rho^{\mathbb{R}^3}(y, x)$.

(30) For all elements x, y, z of $[\![\mathbb{R}, \mathbb{R}, \mathbb{R}]\!]$ such that $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ and $z = \langle z_1, z_2, z_3 \rangle$ holds $\rho^{\mathbb{R}^3}(x, z) \leq \rho^{\mathbb{R}^3}(x, y) + \rho^{\mathbb{R}^3}(y, z)$.

The Euclidean space being a metric space is defined as follows:

(Def.14) the Euclidean space$= \langle [\![\mathbb{R}, \mathbb{R}, \mathbb{R}]\!], \rho^{\mathbb{R}^3} \rangle$.

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