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Metric Spaces as Topological Spaces - Fundamental Concepts

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Summary. Some notions connected with metric spaces and the relationship between metric spaces and topological spaces. Compactness of topological spaces is transferred for the case of metric spaces [13]. Some basic theorems about translations of topological notions for metric spaces are proved. One-dimensional topological space $\mathbb{R}^{\mathbf{1}}$ is introduced, too.

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The papers [21], [11], [1], [22], [20], [4], [5], [6], [12], [10], [3], [14], [16], [23], [9], [7], [2], [15], [18], [17], [19], and [8] provide the notation and terminology for this paper. For simplicity we follow a convention: a, b, r will denote real numbers, n will denote a natural number, T will denote a topological space, and F will denote a family of subsets of T. One can prove the following proposition

(1) F is a cover of T if and only if the carrier of $T \subseteq \bigcup F$.

In the sequel A will be a subspace of T. Next we state three propositions:

(2) For every point p of A holds p is a point of T.

(3) If T is a T_2 space, then A is a T_2 space.

(4) For all subspaces A, B of T such that the carrier of $A \subseteq$ the carrier of B holds A is a subspace of B.

In the sequel P, Q denote subsets of T and p denotes a point of T. We now state several propositions:

(5) If $P \neq \emptyset_T$, then $T \restriction P$ is a subspace of $T \restriction P \cup Q$ **qua** a subset of T but if $Q \neq \emptyset_T$, then $T \restriction Q$ is a subspace of $T \restriction P \cup Q$ **qua** a subset of T.

(6) If $P \neq \emptyset$ and $p \in P$, then for every neighborhood Q of p and for every point p' of $T \restriction P$ and for every subset Q' of $T \restriction P$ such that $Q' = Q \cap P$ and $p' = p$ holds Q' is a neighborhood of p'.

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(7) For all topological spaces A, B, C and for every map f from A into C such that f is continuous and C is a subspace of B for every map h from A into B such that $h = f$ holds h is continuous.

(8) For all topological spaces A, B and for every map f from A into B and for every subspace C of B such that f is continuous and $\operatorname{rng} f \subseteq$ the carrier of C for every map h from A into C such that $h = f$ holds h is continuous.

(9) For all topological spaces A, B and for every map f from A into B and for every subset C of B such that f is continuous and $\operatorname{rng} f \subseteq C$ and $C \neq \emptyset$ for every map h from A into $B \restriction C$ such that $h = f$ holds h is continuous.

(10) For all topological spaces T, S and for every map f from T into S such that f is continuous for every subset P of T and for every map h from $T \restriction P$ into S such that $P \neq \emptyset_T$ and $h = f \restriction P$ holds h is continuous.

In the sequel M will denote a metric space and p will denote a point of M. One can prove the following proposition

(11) If $r > 0$, then $p \in \mathrm{Ball}(p, r)$.

We now define two new modes. Let us consider M. A subset of M is sets of points of M.

A family of subsets of M is a family of subsets of the carrier of M.

Let us consider M. A metric space is said to be a subspace of M if:

(Def.1) the carrier of it $\subseteq$ the carrier of M and for all points x, y of it holds (the distance of it)$(x, y) =$ (the distance of M)(x, y).

In the sequel A will be a subspace of M. One can prove the following propositions:

(12) For every point p of A holds p is a point of M.

(13) For every point x of M and for every point x' of A such that $x = x'$ holds $\mathrm{Ball}(x', r) = \mathrm{Ball}(x, r) \cap$ the carrier of A.

Let M be a metric space, and let A be a non-empty subset of M. The functor $M \restriction A$ yielding a subspace of M is defined as follows:

(Def.2) the carrier of $M \restriction A = A$.

Let us consider a, b. Let us assume that $a \leq b$. The functor $[a,\, b]_{\mathrm{M}}$ yields a subspace of the metric space of real numbers and is defined by:

(Def.3) for every non-empty subset P of the metric space of real numbers such that $P = [a, b]$ holds $[a,\, b]_{\mathrm{M}} =$ (the metric space of real numbers) $\restriction P$.

We now state the proposition

(14) If $a \leq b$, then the carrier of $[a,\, b]_{\mathrm{M}} = [a, b]$.

In the sequel F, G will be families of subsets of M. We now define two new predicates. Let us consider M, F. We say that F is a family of balls if and only if:

(Def.4) for an arbitrary P such that $P \in F$ there exist p, r such that $P = \mathrm{Ball}(p, r)$.

We say that F is a cover of M if and only if:

(Def.5) the carrier of $M \subseteq \bigcup F$.

The following propositions are true:

(15) For all points p, q of the metric space of real numbers and for all real numbers x, y such that $x = p$ and $y = q$ holds $\rho(p, q) = |x - y|$.

(16) The carrier of $M =$ the carrier of $M_{\rm top}$ and the topology of $M_{\rm top} =$ the open set family of M.

(17) For every family F of subsets of M holds F is a family of subsets of $M_{\rm top}$.

(18) For every family F of subsets of $M_{\rm top}$ holds F is a family of subsets of M.

(19) $A_{\rm top}$ is a subspace of $M_{\rm top}$.

(20) For every subset P of $\mathcal{E}^n_{\rm T}$ and for every non-empty subset Q of $\mathcal{E}^n$ such that $P = Q$ holds $(\mathcal{E}^n_{\rm T}) \restriction P = (\mathcal{E}^n \restriction Q)_{\rm top}$.

(21) For every subset P of $M_{\rm top}$ such that $P = \mathrm{Ball}(p, r)$ holds P is open.

(22) For every subset P of $M_{\rm top}$ holds P is open if and only if for every point p of M such that $p \in P$ there exists r such that $r > 0$ and $\mathrm{Ball}(p, r) \subseteq P$.

Let us consider M. We say that M is compact if and only if:

(Def.6) $M_{\rm top}$ is compact.

We now state the proposition

(23) M is compact if and only if for every F such that F is a family of balls and F is a cover of M there exists G such that $G \subseteq F$ and G is a cover of M and G is finite.

The topological space $\mathbb{R}^{\mathbf{1}}$ is defined as follows:

(Def.7) $\mathbb{R}^{\mathbf{1}} =$ (the metric space of real numbers)$_{\rm top}$.

One can prove the following proposition

(24) The carrier of $\mathbb{R}^{\mathbf{1}} = \mathbb{R}$.

Let us consider a, b. Let us assume that $a \leq b$. The functor $[a, b]_{\rm T}$ yields a subspace of $\mathbb{R}^{\mathbf{1}}$ and is defined by:

(Def.8) $[a, b]_{\rm T} = ([a, b]_{\rm M})_{\rm top}$.

We now state three propositions:

(25) If $a \leq b$, then the carrier of $[a, b]_{\rm T} = [a, b]$.

(26) If $a \leq b$, then for every subset P of $\mathbb{R}^{\mathbf{1}}$ such that $P = [a, b]$ holds $[a, b]_{\rm T} = \mathbb{R}^{\mathbf{1}} \restriction P$.

(27) $[0, 1]_{\rm T} = \mathbb{I}$.

Let us note that it makes sense to consider the following constant. Then $\mathbb{I}$ is a subspace of $\mathbb{R}^{\mathbf{1}}$.

The following proposition is true

(28) For every map f from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$ such that there exist real numbers a, b such that for every real number x holds $f(x) = a \cdot x + b$ holds f is continuous.

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