FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

Comma Category

Grzegorz Bancerek
Warsaw University, Białystok
IM PAN, Warszawa

Agata Darmochwał
Warsaw University

Summary. Comma category of two functors is introduced.

MML Identifier: `COMMACAT`.

The terminology and notation used in this paper have been introduced in the following articles: [9], [10], [1], [5], [2], [7], [4], [3], [6], and [8]. We now define four new functors. Let x be arbitrary. The functor $x_{\mathbf{1},\mathbf{1}}$ is defined by:

(Def.1) $x_{\mathbf{1},\mathbf{1}} = (x_{\mathbf{1}})_{\mathbf{1}}$.

The functor $x_{\mathbf{1},\mathbf{2}}$ is defined as follows:

(Def.2) $x_{\mathbf{1},\mathbf{2}} = (x_{\mathbf{1}})_{\mathbf{2}}$.

The functor $x_{\mathbf{2},\mathbf{1}}$ is defined by:

(Def.3) $x_{\mathbf{2},\mathbf{1}} = (x_{\mathbf{2}})_{\mathbf{1}}$.

The functor $x_{\mathbf{2},\mathbf{2}}$ is defined as follows:

(Def.4) $x_{\mathbf{2},\mathbf{2}} = (x_{\mathbf{2}})_{\mathbf{2}}$.

In the sequel x, x_1, x_2, y, y_1, y_2 are arbitrary. One can prove the following proposition

(1) $\langle\langle x_1, x_2\rangle, y\rangle_{\mathbf{1},\mathbf{1}} = x_1$ and $\langle\langle x_1, x_2\rangle, y\rangle_{\mathbf{1},\mathbf{2}} = x_2$ and $\langle x, \langle y_1, y_2\rangle\rangle_{\mathbf{2},\mathbf{1}} = y_1$ and $\langle x, \langle y_1, y_2\rangle\rangle_{\mathbf{2},\mathbf{2}} = y_2$.

Let D_1, D_2, D_3 be non-empty sets, and let x be an element of $[\!:[\!: D_1, D_2 :\!], D_3 :\!]$. Then $x_{\mathbf{1},\mathbf{1}}$ is an element of D_1. Then $x_{\mathbf{1},\mathbf{2}}$ is an element of D_2.

Let D_1, D_2, D_3 be non-empty sets, and let x be an element of $[\!: D_1, [\!: D_2, D_3 :\!]:\!]$. Then $x_{\mathbf{2},\mathbf{1}}$ is an element of D_2. Then $x_{\mathbf{2},\mathbf{2}}$ is an element of D_3.

For simplicity we follow a convention: C, D, E are categories, c is an object of C, d is an object of D, x is arbitrary, f is a morphism of E, g is a morphism of C, h is a morphism of D, F is a functor from C to E, and G is a functor from D to E. Let us consider C, D, E, and let F be a functor from C to E, and let G be a functor from D to E. Let us assume that there exist c_1, d_1, f_1 such

ISSN 0777–4028

that $f_1 \in \hom(F(c_1), G(d_1))$. The functor $\mathrm{Obj}_{(F,G)}$ yields a non-empty subset of $[\![[\![$ the objects of C, the objects of $D]\!]$, the morphisms of $E]\!]$ and is defined as follows:

(Def.5) $\mathrm{Obj}_{(F,G)} = \{\langle\langle c, d\rangle, f\rangle : f \in \hom(F(c), G(d))\}$.

In the sequel o, o_1, o_2 will denote elements of $\mathrm{Obj}_{(F,G)}$. The following proposition is true

(2) Suppose there exist c, d, f such that $f \in \hom(F(c), G(d))$. Then $o = \langle\langle o_{\mathbf{1},\mathbf{1}}, o_{\mathbf{1},\mathbf{2}}\rangle, o_{\mathbf{2}}\rangle$ and $o_{\mathbf{2}} \in \hom(F(o_{\mathbf{1},\mathbf{1}}), G(o_{\mathbf{1},\mathbf{2}}))$ and $\mathrm{dom}(o_{\mathbf{2}}) = F(o_{\mathbf{1},\mathbf{1}})$ and $\mathrm{cod}(o_{\mathbf{2}}) = G(o_{\mathbf{1},\mathbf{2}})$.

Let us consider C, D, E, F, G. Let us assume that there exist c_1, d_1, f_1 such that $f_1 \in \hom(F(c_1), G(d_1))$. The functor $\mathrm{Morph}_{(F,G)}$ yielding a non-empty subset of $[\![[\![\mathrm{Obj}_{(F,G)}, \mathrm{Obj}_{(F,G)}$ **qua** a non-empty set $]\!]$, $[\![$ the morphisms of C, the morphisms of $D]\!]]\!]$ is defined by:

(Def.6) $\mathrm{Morph}_{(F,G)} = \{\langle\langle o_1, o_2\rangle, \langle g, h\rangle\rangle : \mathrm{dom}\, g = o_{1\mathbf{1},\mathbf{1}} \wedge \mathrm{cod}\, g = o_{2\mathbf{1},\mathbf{1}} \wedge \mathrm{dom}\, h = o_{1\mathbf{1},\mathbf{2}} \wedge \mathrm{cod}\, h = o_{2\mathbf{1},\mathbf{2}} \wedge o_{2\mathbf{2}} \cdot F(g) = G(h) \cdot o_{1\mathbf{2}}\}$.

In the sequel k, k_1, k_2, k' denote elements of $\mathrm{Morph}_{(F,G)}$. Let us consider C, D, E, F, G, k. Then $k_{\mathbf{1},\mathbf{1}}$ is an element of $\mathrm{Obj}_{(F,G)}$. Then $k_{\mathbf{1},\mathbf{2}}$ is an element of $\mathrm{Obj}_{(F,G)}$. Then $k_{\mathbf{2},\mathbf{1}}$ is a morphism of C. Then $k_{\mathbf{2},\mathbf{2}}$ is a morphism of D.

The following proposition is true

(3) Suppose There exist c, d, f such that $f \in \hom(F(c), G(d))$. Then
(i) $k = \langle\langle k_{\mathbf{1},\mathbf{1}}, k_{\mathbf{1},\mathbf{2}}\rangle, \langle k_{\mathbf{2},\mathbf{1}}, k_{\mathbf{2},\mathbf{2}}\rangle\rangle$,
(ii) $\mathrm{dom}(k_{\mathbf{2},\mathbf{1}}) = (k_{\mathbf{1},\mathbf{1}})_{\mathbf{1},\mathbf{1}}$,
(iii) $\mathrm{cod}(k_{\mathbf{2},\mathbf{1}}) = (k_{\mathbf{1},\mathbf{2}})_{\mathbf{1},\mathbf{1}}$,
(iv) $\mathrm{dom}(k_{\mathbf{2},\mathbf{2}}) = (k_{\mathbf{1},\mathbf{1}})_{\mathbf{1},\mathbf{2}}$,
(v) $\mathrm{cod}(k_{\mathbf{2},\mathbf{2}}) = (k_{\mathbf{1},\mathbf{2}})_{\mathbf{1},\mathbf{2}}$,
(vi) $(k_{\mathbf{1},\mathbf{2}})_{\mathbf{2}} \cdot F(k_{\mathbf{2},\mathbf{1}}) = G(k_{\mathbf{2},\mathbf{2}}) \cdot (k_{\mathbf{1},\mathbf{1}})_{\mathbf{2}}$.

Let us consider C, D, E, F, G, k_1, k_2. Let us assume that there exist c_1, d_1, f_1 such that $f_1 \in \hom(F(c_1), G(d_1))$. Let us assume that $k_{1\mathbf{1},\mathbf{2}} = k_{2\mathbf{1},\mathbf{1}}$. The functor $k_2 \cdot k_1$ yielding an element of $\mathrm{Morph}_{(F,G)}$ is defined as follows:

(Def.7) $k_2 \cdot k_1 = \langle\langle k_{1\mathbf{1},\mathbf{1}}, k_{2\mathbf{1},\mathbf{2}}\rangle, \langle k_{2\mathbf{2},\mathbf{1}} \cdot k_{1\mathbf{2},\mathbf{1}}, k_{2\mathbf{2},\mathbf{2}} \cdot k_{1\mathbf{2},\mathbf{2}}\rangle\rangle$.

Let us consider C, D, E, F, G. The functor $\circ_{(F,G)}$ yields a partial function from $[\![\mathrm{Morph}_{(F,G)}, \mathrm{Morph}_{(F,G)}]\!]$ to $\mathrm{Morph}_{(F,G)}$ and is defined by:

(Def.8) $\mathrm{dom}(\circ_{(F,G)}) = \{\langle k_1, k_2\rangle : k_{1\mathbf{1},\mathbf{1}} = k_{2\mathbf{1},\mathbf{2}}\}$ and for all k, k' such that $\langle k, k'\rangle \in \mathrm{dom}(\circ_{(F,G)})$ holds $\circ_{(F,G)}(\langle k, k'\rangle) = k \cdot k'$.

Let us consider C, D, E, F, G. Let us assume that there exist c_1, d_1, f_1 such that $f_1 \in \hom(F(c_1), G(d_1))$. The functor (F, G) yielding a strict category is defined by the conditions (Def.9).

(Def.9) (i) The objects of $(F, G) = \mathrm{Obj}_{(F,G)}$,
(ii) the morphisms of $(F, G) = \mathrm{Morph}_{(F,G)}$,
(iii) for every k holds (the dom-map of (F, G))$(k) = k_{\mathbf{1},\mathbf{1}}$,
(iv) for every k holds (the cod-map of (F, G))$(k) = k_{\mathbf{1},\mathbf{2}}$,
(v) for every o holds (the id-map of (F, G))$(o) = \langle\langle o, o\rangle, \langle \mathrm{id}_{(o_{\mathbf{1},\mathbf{1}})}, \mathrm{id}_{(o_{\mathbf{1},\mathbf{2}})}\rangle\rangle$,

(vi) the composition of $(F,G) = \circ_{(F,G)}$.

We now state two propositions:

(4) The objects of $\dot{\circlearrowleft}(x,y) = \{x\}$ and the morphisms of $\dot{\circlearrowleft}(x,y) = \{y\}$.

(5) For all objects a, b of $\dot{\circlearrowleft}(x,y)$ holds $\hom(a,b) = \{y\}$.

Let us consider C, c. The functor $\dot{\circlearrowleft}(c)$ yielding a strict subcategory of C is defined as follows:

(Def.10) $\dot{\circlearrowleft}(c) = \dot{\circlearrowleft}(c, \mathrm{id}_c)$.

We now define two new functors. Let us consider C, c. The functor (c,C) yields a strict category and is defined by:

(Def.11) $(c,C) = (\overset{\dot{\circlearrowleft}(c)}{\hookrightarrow}, \mathrm{id}_C)$.

The functor (C,c) yields a strict category and is defined as follows:

(Def.12) $(C,c) = (\mathrm{id}_C, \overset{\dot{\circlearrowleft}(c)}{\hookrightarrow})$.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[3] Czesław Byliński. Introduction to categories and functors. *Formalized Mathematics*, 1(**2**):409–420, 1990.

[4] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[6] Czesław Byliński. Subcategories and products of categories. *Formalized Mathematics*, 1(**4**):725–732, 1990.

[7] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[8] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[10] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

Received February 20, 1992