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Isomorphisms of Categories

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Summary. We continue the development of the category theory basically following [12] (compare also [11]). We define the concept of isomorphic categories and prove basic facts related, e.g. that the Cartesian product of categories is associative up to the isomorphism. We introduce the composition of a functor and a transformation, and of transformation and a functor, and afterwards we define again those concepts for natural transformations. Let us observe, that we have to duplicate those concepts because of the permissiveness: if a functor F is not naturally transformable to G, then natural transformation from F to G has no fixed meaning, hence we cannot claim that the composition of it with a functor as a transformation results in a natural transformation. We define also the so called horizontal composition of transformations ([12], p.140, exercise **4.2**,**5**(C)) and prove *interchange law* ([11], p.44). We conclude with the definition of equivalent categories.

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The articles [16], [17], [4], [5], [3], [7], [1], [2], [10], [13], [8], [14], [6], [9], and [15] provide the notation and terminology for this paper. We adopt the following convention: A, B, C, D will denote categories, F, F_1, F_2 will denote functors from A to B, and G will denote a functor from B to C. One can prove the following propositions:

(1) For all functions F, G such that F is one-to-one and G is one-to-one holds $[\!: F, G :\!]$ is one-to-one.

(2) $\operatorname{rng} \pi_1(A \times B) =$ the morphisms of A and $\operatorname{rng} \pi_2(B \times A) =$ the morphisms of A.

(3) For every morphism f of A such that f is invertible holds $F(f)$ is invertible.

(4) For every functor F from A to B and for every functor G from B to A holds $F \cdot \mathrm{id}_A = F$ and $\mathrm{id}_A \cdot G = G$.

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(5) For all objects a, b of A such that $\hom(a,b) \neq \emptyset$ and for every morphism f from a to b and for every functor F from A to B and for every functor G from B to C holds $(G \cdot F)(f) = G(F(f))$.

(6) For all objects a, b, c of A such that $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and for every morphism f from a to b and for every morphism g from b to c and for every functor F from A to B holds $F(g \cdot f) = F(g) \cdot F(f)$.

(7) For all functors F_1, F_2 from A to B such that F_1 is transformable to F_2 and for every transformation t from F_1 to F_2 and for every object a of A holds $t(a) \in \hom(F_1(a), F_2(a))$.

(8) For all functors F_1, F_2 from A to B and for all functors G_1, G_2 from B to C such that F_1 is transformable to F_2 and G_1 is transformable to G_2 holds $G_1 \cdot F_1$ is transformable to $G_2 \cdot F_2$.

(9) For all functors F_1, F_2 from A to B such that F_1 is transformable to F_2 and for every transformation t from F_1 to F_2 such that t is invertible and for every object a of A holds $F_1(a)$ and $F_2(a)$ are isomorphic.

Let us consider C, D. Let us observe that the mode below can be characterized by another conditions, which are equivalent to the formulas previously defining them. In accordance the mode Let us note that one can characterize the mode functor from C to D, by the following (equivalent) condition:

(Def.1) (i) for every object c of C there exists an object d of D such that $\mathrm{it}(\mathrm{id}_c) = \mathrm{id}_d$,

(ii) for every morphism f of C holds $\mathrm{it}(\mathrm{id}_{\mathrm{dom}\, f}) = \mathrm{id}_{\mathrm{dom}\,\mathrm{it}(f)}$ and $\mathrm{it}(\mathrm{id}_{\mathrm{cod}\, f}) = \mathrm{id}_{\mathrm{cod}\,\mathrm{it}(f)}$,

(iii) for all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $\mathrm{it}(g \cdot f) = \mathrm{it}(g) \cdot \mathrm{it}(f)$.

Let us consider A. Then id_A is a functor from A to A. Let us consider B, C, and let F be a functor from A to B, and let G be a functor from B to C. Then $G \cdot F$ is a functor from A to C.

In the sequel o, m are arbitrary. We now state three propositions:

(10) If F is an isomorphism, then for every morphism g of B there exists a morphism f of A such that $F(f) = g$.

(11) If F is an isomorphism, then for every object b of B there exists an object a of A such that $F(a) = b$.

(12) If F is one-to-one, then $\mathrm{Obj}\, F$ is one-to-one.

Let us consider A, B, F. Let us assume that F is an isomorphism. The functor F^{-1} yields a functor from B to A and is defined by:

(Def.2) $F^{-1} = F^{-1}$.

Let us consider A, B, F. Let us note that one can characterize the predicate F is an isomorphism by the following (equivalent) condition:

(Def.3) F is one-to-one and $\mathrm{rng}\, F =$ the morphisms of B.

Next we state several propositions:

(13) If F is an isomorphism, then F^{-1} is an isomorphism.

(14) If F is an isomorphism, then $(\mathrm{Obj}\, F)^{-1} = \mathrm{Obj}(F^{-1})$.

(15) If F is an isomorphism, then $(F^{-1})^{-1} = F$.

(16) If F is an isomorphism, then $F \cdot F^{-1} = \mathrm{id}_B$ and $F^{-1} \cdot F = \mathrm{id}_A$.

(17) If F is an isomorphism and G is an isomorphism, then $G \cdot F$ is an isomorphism.

In the sequel t_1 denotes a natural transformation from F_1 to F_2 and t_2 denotes a natural transformation from F to F_2. We now define two new predicates. Let us consider A, B. We say that A and B are isomorphic if and only if:

(Def.4) there exists a functor F from A to B such that F is an isomorphism.

We write $A \cong B$ if A and B are isomorphic.

The following propositions are true:

(18) $A \cong A$.

(19) If $A \cong B$, then $B \cong A$.

(20) If $A \cong B$ and $B \cong C$, then $A \cong C$.

(21) $[: \dot{\circlearrowleft}(o, m), A :] \cong A$.

(22) $[: A, B :] \cong [: B, A :]$.

(23) $[: [: A, B :], C :] \cong [: A, [: B, C :] :]$.

(24) If $A \cong B$ and $C \cong D$, then $[: A, C :] \cong [: B, D :]$.

Let us consider A, B, C, and let F_1, F_2 be functors from A to B satisfying the condition: F_1 is transformable to F_2. Let t be a transformation from F_1 to F_2, and let G be a functor from B to C. The functor $G \cdot t$ yields a transformation from $G \cdot F_1$ to $G \cdot F_2$ and is defined as follows:

(Def.5) $G \cdot t = G \cdot t$.

Let us consider A, B, C, and let G_1, G_2 be functors from B to C satisfying the condition: G_1 is transformable to G_2. Let F be a functor from A to B, and let t be a transformation from G_1 to G_2. The functor $t \cdot F$ yielding a transformation from $G_1 \cdot F$ to $G_2 \cdot F$ is defined by:

(Def.6) $t \cdot F = t \cdot \mathrm{Obj}\, F$.

We now state three propositions:

(25) For all functors G_1, G_2 from B to C such that G_1 is transformable to G_2 and for every functor F from A to B and for every transformation t from G_1 to G_2 and for every object a of A holds $(t \cdot F)(a) = t(F(a))$.

(26) For all functors F_1, F_2 from A to B such that F_1 is transformable to F_2 and for every transformation t from F_1 to F_2 and for every functor G from B to C and for every object a of A holds $(G \cdot t)(a) = G(t(a))$.

(27) For all functors F_1, F_2 from A to B and for all functors G_1, G_2 from B to C such that F_1 is naturally transformable to F_2 and G_1 is naturally transformable to G_2 holds $G_1 \cdot F_1$ is naturally transformable to $G_2 \cdot F_2$.

Let us consider A, B, C, and let F_1, F_2 be functors from A to B satisfying the condition: F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2, and let G be a functor from B to C. The functor $G \cdot t$ yielding a natural transformation from $G \cdot F_1$ to $G \cdot F_2$ is defined by:

(Def.7) $G \cdot t = G \cdot t$.

Next we state the proposition

(28) For all functors F_1, F_2 from A to B such that F_1 is naturally transformable to F_2 and for every natural transformation t from F_1 to F_2 and for every functor G from B to C and for every object a of A holds $(G \cdot t)(a) = G(t(a))$.

Let us consider A, B, C, and let G_1, G_2 be functors from B to C satisfying the condition: G_1 is naturally transformable to G_2. Let F be a functor from A to B, and let t be a natural transformation from G_1 to G_2. The functor $t \cdot F$ yields a natural transformation from $G_1 \cdot F$ to $G_2 \cdot F$ and is defined as follows:

(Def.8) $t \cdot F = t \cdot F$.

The following proposition is true

(29) For all functors G_1, G_2 from B to C such that G_1 is naturally transformable to G_2 and for every functor F from A to B and for every natural transformation t from G_1 to G_2 and for every object a of A holds $(t \cdot F)(a) = t(F(a))$.

For simplicity we follow the rules: F, F_1, F_2, F_3 are functors from A to B, G, G_1, G_2, G_3 are functors from B to C, H, H_1, H_2 are functors from C to D, s is a natural transformation from F_1 to F_2, s' is a natural transformation from F_2 to F_3, t is a natural transformation from G_1 to G_2, t' is a natural transformation from G_2 to G_3, and u is a natural transformation from H_1 to H_2. We now state a number of propositions:

(30) If F_1 is naturally transformable to F_2, then for every object a of A holds $\hom(F_1(a), F_2(a)) \neq \emptyset$.

(31) If F_1 is naturally transformable to F_2, then for all natural transformations t_1, t_2 from F_1 to F_2 such that for every object a of A holds $t_1(a) = t_2(a)$ holds $t_1 = t_2$.

(32) If F_1 is naturally transformable to F_2 and F_2 is naturally transformable to F_3, then $G \cdot (s' \circ s) = G \cdot s' \circ G \cdot s$.

(33) If G_1 is naturally transformable to G_2 and G_2 is naturally transformable to G_3, then $(t' \circ t) \cdot F = t' \cdot F \circ t \cdot F$.

(34) If H_1 is naturally transformable to H_2, then $(u \cdot G) \cdot F = u \cdot (G \cdot F)$.

(35) If G_1 is naturally transformable to G_2, then $(H \cdot t) \cdot F = H \cdot (t \cdot F)$.

(36) If F_1 is naturally transformable to F_2, then $(H \cdot G) \cdot s = H \cdot (G \cdot s)$.

(37) $\mathrm{id}_G \cdot F = \mathrm{id}_{(G \cdot F)}$.

(38) $G \cdot \mathrm{id}_F = \mathrm{id}_{(G \cdot F)}$.

(39) If G_1 is naturally transformable to G_2, then $t \cdot \mathrm{id}_B = t$.

(40) If F_1 is naturally transformable to F_2, then $\mathrm{id}_B \cdot s = s$.

Let us consider A, B, C, F_1, F_2, G_1, G_2, s, t. The functor $t\,s$ yields a natural transformation from $G_1 \cdot F_1$ to $G_2 \cdot F_2$ and is defined as follows:

(Def.9) $t\,s = t \cdot F_2 \circ G_1 \cdot s$.

We now state several propositions:

(41) If F_1 is naturally transformable to F_2 and G_1 is naturally transformable to G_2, then $t\,s = G_2 \cdot s \circ t \cdot F_1$.

(42) If F_1 is naturally transformable to F_2, then $\mathrm{id}_{(\mathrm{id}_B)}\,s = s$.

(43) If G_1 is naturally transformable to G_2, then $t\,\mathrm{id}_{(\mathrm{id}_B)} = t$.

(44) If F_1 is naturally transformable to F_2 and G_1 is naturally transformable to G_2 and H_1 is naturally transformable to H_2, then $u\,(t\,s) = (u\,t)\,s$.

(45) If G_1 is naturally transformable to G_2, then $t \cdot F = t\,\mathrm{id}_F$.

(46) If F_1 is naturally transformable to F_2, then $G \cdot s = \mathrm{id}_G\,s$.

(47) If F_1 is naturally transformable to F_2 and F_2 is naturally transformable to F_3 and G_1 is naturally transformable to G_2 and G_2 is naturally transformable to G_3, then $(t' \circ t)\,(s' \circ s) = t'\,s' \circ t\,s$.

(48) For every functor F from A to B and for every functor G from C to D and for all functors I, J from B to C such that $I \cong J$ holds $G \cdot I \cong G \cdot J$ and $I \cdot F \cong J \cdot F$.

(49) For every functor F from A to B and for every functor G from B to A and for every functor I from A to A such that $I \cong \mathrm{id}_A$ holds $F \cdot I \cong F$ and $I \cdot G \cong G$.

We now define two new predicates. Let A, B be categories. We say that A is equivalent with B if and only if:

(Def.10) there exists a functor F from A to B and there exists a functor G from B to A such that $G \cdot F \cong \mathrm{id}_A$ and $F \cdot G \cong \mathrm{id}_B$.

A and B are equivalent stands for A is equivalent with B.

We now state four propositions:

(50) If $A \cong B$, then A is equivalent with B.

(51) A is equivalent with A.

(52) If A and B are equivalent, then B and A are equivalent.

(53) If A and B are equivalent and B and C are equivalent, then A and C are equivalent.

Let us consider A, B. Let us assume that A and B are equivalent. A functor from A to B is called an equivalence of A and B if:

(Def.11) there exists a functor G from B to A such that $G \cdot \mathrm{it} \cong \mathrm{id}_A$ and $\mathrm{it} \cdot G \cong \mathrm{id}_B$.

Next we state several propositions:

(54) id_A is an equivalence of A and A.

(55) If A and B are equivalent and B and C are equivalent, then for every equivalence F of A and B and for every equivalence G of B and C holds $G \cdot F$ is an equivalence of A and C.

(56) If A and B are equivalent, then for every equivalence F of A and B there exists an equivalence G of B and A such that $G \cdot F \cong \mathrm{id}_A$ and $F \cdot G \cong \mathrm{id}_B$.

(57) For every functor F from A to B and for every functor G from B to A such that $G \cdot F \cong \mathrm{id}_A$ holds F is faithful.

(58) If A and B are equivalent, then for every equivalence F of A and B holds F is full and F is faithful and for every object b of B there exists an object a of A such that b and $F(a)$ are isomorphic.

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