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# Transpose Matrices and Groups of Permutations

Katarzyna Jankowska
Warsaw University
Białystok

**Summary.** Some facts concerning matrices with dimention $2 \times 2$ are shown. Upper and lower triangular matrices, and operation of deleting rows and columns in a matrix are introduced. Besides, we deal with sets of permutations and the fact that all permutations of finite set constitute a finite group is proved. Some proofs are based on [11] and [14].

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The articles [17], [7], [8], [3], [15], [2], [1], [19], [18], [21], [20], [4], [13], [16], [9], [6], [12], [10], and [5] provide the notation and terminology for this paper.

## 1. Some examples of matrices

For simplicity we follow a convention: $x$, $x_1$, $x_2$, $y_1$, $y_2$ are arbitrary, $i$, $j$, $k$, $n$, $m$ are natural numbers, $D$ is a non-empty set, $K$ is a field, $s$ is a finite sequence, and $a$, $b$, $c$, $d$ are elements of $D$. The scheme *SeqDEx* concerns a non-empty set $\mathcal{A}$, a natural number $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a finite sequence $p$ of elements of $\mathcal{A}$ such that $\operatorname{dom} p = \operatorname{Seg} \mathcal{B}$ and for every $k$ such that $k \in \operatorname{Seg} \mathcal{B}$ holds $\mathcal{P}[k, p(k)]$

provided the following requirement is met:

- for every $k$ such that $k \in \operatorname{Seg} \mathcal{B}$ there exists an element $x$ of $\mathcal{A}$ such that $\mathcal{P}[k, x]$.

Let us consider $D$, $a$, $b$. Then $\langle a, b \rangle$ is a finite sequence of elements of $D$.

Let us consider $n$, $m$, and let $a$ be arbitrary. The functor $\begin{pmatrix} a & \ldots & a \\ \vdots & \ddots & \vdots \\ a & \ldots & a \end{pmatrix}^{n \times m}$ yielding a tabular finite sequence is defined as follows:




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(Def.1) $$\begin{pmatrix} a & \ldots & a \\ \vdots & \ddots & \vdots \\ a & \ldots & a \end{pmatrix}^{n \times m} = n \longmapsto (m \longmapsto a).$$

Let us consider $D$, $n$, $m$, $d$. Then $\begin{pmatrix} d & \ldots & d \\ \vdots & \ddots & \vdots \\ d & \ldots & d \end{pmatrix}^{n \times m}$ is a matrix over $D$ of dimension $n \times m$.

Next we state the proposition

(1) If $\langle i, j \rangle \in$ the indices of $\begin{pmatrix} a & \ldots & a \\ \vdots & \ddots & \vdots \\ a & \ldots & a \end{pmatrix}^{n \times m}$, then $(\begin{pmatrix} a & \ldots & a \\ \vdots & \ddots & \vdots \\ a & \ldots & a \end{pmatrix}^{n \times m})_{i,j} = a$.

In the sequel $a'$, $b'$ are elements of the carrier of $K$. Next we state the proposition

(2) $$\begin{pmatrix} a' & \ldots & a' \\ \vdots & \ddots & \vdots \\ a' & \ldots & a' \end{pmatrix}^{n \times n} + \begin{pmatrix} b' & \ldots & b' \\ \vdots & \ddots & \vdots \\ b' & \ldots & b' \end{pmatrix}^{n \times n} = \begin{pmatrix} a'+b' & \ldots & a'+b' \\ \vdots & \ddots & \vdots \\ a'+b' & \ldots & a'+b' \end{pmatrix}^{n \times n}.$$

Let $a$, $b$, $c$, $d$ be arbitrary. The functor $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ yielding a tabular finite sequence is defined as follows:

(Def.2) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \langle \langle a, b \rangle, \langle c, d \rangle \rangle.$$

The following two propositions are true:

(3) len $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = 2$ and width $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = 2$ and the indices of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = [\!:\mathrm{Seg}\,2, \mathrm{Seg}\,2:\!]$.

(4) $\langle 1, 1 \rangle \in$ the indices of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ and $\langle 1, 2 \rangle \in$ the indices of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ and $\langle 2, 1 \rangle \in$ the indices of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ and $\langle 2, 2 \rangle \in$ the indices of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$.

Let us consider $D$, and let $a$ be an element of $D$. Then $\langle a \rangle$ is an element of $D^1$.

Let us consider $D$, and let us consider $n$, and let $p$ be an element of $D^n$. Then $\langle p \rangle$ is a matrix over $D$ of dimension $1 \times n$.

One can prove the following proposition

(5) $\langle 1, 1 \rangle \in$ the indices of $\langle\langle a \rangle\rangle$ and $\langle\langle a \rangle\rangle_{1,1} = a$.

Let us consider $D$, and let $a$, $b$, $c$, $d$ be elements of $D$. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix over $D$ of dimension 2.

Next we state the proposition

(6) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{1,1} = a$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{1,2} = b$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2,1} = c$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2,2} = d$.

Let us consider $n$, and let $K$ be a field. A matrix over $K$ of dimension $n$ is said to be an upper triangular matrix over $K$ of dimension $n$ if:

(Def.3) for all $i$, $j$ such that $\langle i, j \rangle \in$ the indices of it holds if $i > j$, then $\mathrm{it}_{i,j} = 0_K$.

Let us consider $n$, $K$. A matrix over $K$ of dimension $n$ is said to be a lower triangular matrix over $K$ of dimension $n$ if:

(Def.4) for all $i$, $j$ such that $\langle i, j \rangle \in$ the indices of it holds if $i < j$, then $\mathrm{it}_{i,j} = 0_K$.

The following proposition is true

(7) For every matrix $M$ over $D$ such that $\operatorname{len} M = n$ holds $M$ is a matrix over $D$ of dimension $n \times \operatorname{width} M$.

## 2. Deleting of rows and columns in a matrix

Let us consider $i$, and let $p$ be a finite sequence. Let us assume that $i \in \operatorname{dom} p$. The functor $p_{\restriction i}$ yielding a finite sequence is defined by:

(Def.5) $p_{\restriction i} = p \cdot \operatorname{Sgm}(\operatorname{Seg} \operatorname{len} p \setminus \{i\})$.

We now state three propositions:

(8) For every finite sequence $p$ such that $\operatorname{len} p > 0$ and for every $i$ such that $i \in \operatorname{dom} p$ there exists $m$ such that $\operatorname{len} p = m + 1$ and $\operatorname{len}(p_{\restriction i}) = m$.

(9) For every finite sequence $p$ of elements of $D$ and for every $i$ such that $i \in \operatorname{dom} p$ holds $p_{\restriction i}$ is a finite sequence of elements of $D$.

(10) For every matrix $M$ over $K$ of dimension $n \times m$ and for every $k$ such that $k \in \operatorname{Seg} n$ holds $M(k) = \operatorname{Line}(M, k)$.

Let us consider $i$, and let us consider $K$, and let $M$ be a matrix over $K$. Let us assume that $i \in \operatorname{Seg} \operatorname{width} M$. The deleting of $i$-column in $M$ yielding a matrix over $K$ is defined as follows:

(Def.6) len(the deleting of $i$-column in $M$) $= \operatorname{len} M$ and for every $k$ such that $k \in \operatorname{Seg} \operatorname{len} M$ holds (the deleting of $i$-column in $M$)$(k) = \operatorname{Line}(M, k)_{\restriction i}$.

The following propositions are true:

(11) For all matrices $M_1$, $M_2$ over $D$ holds $M_1 = M_2$ if and only if ${M_1}^{\mathrm{T}} = {M_2}^{\mathrm{T}}$ and $\operatorname{len} M_1 = \operatorname{len} M_2$.

(12) For every matrix $M$ over $D$ such that $\operatorname{width} M > 0$ holds $\operatorname{len}(M^{\mathrm{T}}) = \operatorname{width} M$ and $\operatorname{width}(M^{\mathrm{T}}) = \operatorname{len} M$.

(13) For all matrices $M_1$, $M_2$ over $D$ such that $\operatorname{width} M_1 > 0$ and $\operatorname{width} M_2 > 0$ holds $M_1 = M_2$ if and only if ${M_1}^{\mathrm{T}} = {M_2}^{\mathrm{T}}$ and $\operatorname{width}({M_1}^{\mathrm{T}}) = \operatorname{width}({M_2}^{\mathrm{T}})$.

(14) For all matrices $M_1$, $M_2$ over $D$ such that $\operatorname{width} M_1 > 0$ and $\operatorname{width} M_2 > 0$ holds $M_1 = M_2$ if and only if ${M_1}^{\mathrm{T}} = {M_2}^{\mathrm{T}}$ and $\operatorname{width} M_1 = \operatorname{width} M_2$.

(15) For every matrix $M$ over $D$ such that $\operatorname{len} M > 0$ and $\operatorname{width} M > 0$ holds $(M^{\mathrm{T}})^{\mathrm{T}} = M$.

(16) For every matrix $M$ over $D$ and for every $i$ such that $i \in \operatorname{Seg} \operatorname{len} M$ holds $\operatorname{Line}(M, i) = (M^{\mathrm{T}})_{\Box, i}$.

(17) For every matrix $M$ over $D$ and for every $j$ such that $j \in \operatorname{Seg} \operatorname{width} M$ holds $\operatorname{Line}(M^{\mathrm{T}}, j) = M_{\Box, j}$.

(18) For every matrix $M$ over $D$ and for every $i$ such that $i \in \operatorname{Seg} \operatorname{len} M$ holds $M(i) = \operatorname{Line}(M, i)$.

Let us consider $i$, and let us consider $K$, and let $M$ be a matrix over $K$. Let us assume that $i \in \operatorname{Seg} \operatorname{len} M$ and $\operatorname{width} M > 0$. The deleting of $i$-row in $M$ yields a matrix over $K$ and is defined by:

(Def.7) (i) the deleting of $i$-row in $M = \varepsilon$ if $\operatorname{len} M = 1$,

(ii) width(the deleting of $i$-row in $M$) $= \operatorname{width} M$ and for every $k$ such that $k \in \operatorname{Seg} \operatorname{width} M$ holds (the deleting of $i$-row in $M)_{\Box, k} = (M_{\Box, k})_{\restriction i}$, otherwise.

Let us consider $i$, $j$, and let us consider $n$, and let us consider $K$, and let $M$ be a matrix over $K$ of dimension $n$. The deleting of $i$-row and $j$-column in $M$ yields a matrix over $K$ and is defined as follows:

(Def.8) (i) the deleting of $i$-row and $j$-column in $M = \varepsilon$ if $n = 1$,

(ii) the deleting of $i$-row and $j$-column in $M$ = the deleting of $j$-column in the deleting of $i$-row in $M$, otherwise.

## 3. Sets of permutations

Let us consider $n$, and let $q$, $p$ be permutations of $\operatorname{Seg} n$. Then $p \cdot q$ is a permutation of $\operatorname{Seg} n$.

A set is permutational if:

(Def.9) there exists $n$ such that for every $x$ such that $x \in$ it holds $x$ is a permutation of $\operatorname{Seg} n$.

Let $P$ be a permutational non-empty set. The functor $\operatorname{len} P$ yielding a natural number is defined as follows:

(Def.10) there exists $s$ such that $s \in P$ and $\mathrm{len}\, P = \mathrm{len}\, s$.

Let $P$ be a permutational non-empty set. We see that the element of $P$ is a permutation of $\mathrm{Seg}\,\mathrm{len}\, P$.

One can prove the following proposition

(19) For every $n$ there exists a permutational non-empty set $P$ such that $\mathrm{len}\, P = n$.

Let us consider $n$. The permutations of $n$-element set constitute a permutational non-empty set defined as follows:

(Def.11) $x \in$ the permutations of $n$-element set if and only if $x$ is a permutation of $\mathrm{Seg}\, n$.

The following propositions are true:

(20) len(the permutations of $n$-element set) $= n$.

(21) The permutations of 1-element set $= \{\mathrm{id}_1\}$.

Let us consider $n$, and let $p$ be an element of the permutations of $n$-element set. The functor $\mathrm{len}\, p$ yields a natural number and is defined as follows:

(Def.12) there exists a finite sequence $s$ such that $s = p$ and $\mathrm{len}\, p = \mathrm{len}\, s$.

We now state the proposition

(22) For every element $p$ of the permutations of $n$-element set holds $\mathrm{len}\, p = n$.

## 4. Group of permutations

In the sequel $p$, $q$ denote elements of the permutations of $n$-element set. Let us consider $n$. The functor $A_n$ yielding a strict half group structure is defined by:

(Def.13) the carrier of $A_n =$ the permutations of $n$-element set and for all elements $q$, $p$ of the permutations of $n$-element set holds (the operation of $A_n)(q, p) = p \cdot q$.

One can prove the following propositions:

(23) $\mathrm{id}_n$ is an element of $A_n$.

(24) $p \cdot \mathrm{id}_n = p$ and $\mathrm{id}_n \cdot p = p$.

(25) $p \cdot p^{-1} = \mathrm{id}_n$ and $p^{-1} \cdot p = \mathrm{id}_n$.

(26) $p^{-1}$ is an element of $A_n$.

(27) $p$ is an element of $A_n$ if and only if $p$ is an element of the permutations of $n$-element set.

Let us consider $n$. A permutation of $n$ element set is an element of the permutations of $n$-element set.

Then $A_n$ is a strict group.

We now state the proposition

(28) $\mathrm{id}_n = 1_{A_n}$.

Let us consider $n$, and let $p$ be a permutation of $\mathrm{Seg}\, n$. We say that $p$ is a transposition if and only if:

(Def.14) there exist $i$, $j$ such that $i \in \operatorname{dom} p$ and $j \in \operatorname{dom} p$ and $i \neq j$ and $p(i) = j$ and $p(j) = i$ and for every $k$ such that $k \neq i$ and $k \neq j$ and $k \in \operatorname{dom} p$ holds $p(k) = k$.

We now define two new predicates. Let us consider $n$, and let $p$ be a permutation of $\operatorname{Seg} n$. We say that $p$ is even if and only if:

(Def.15) there exists a finite sequence $l$ of elements of the carrier of $A_n$ such that $\operatorname{len} l \bmod 2 = 0$ and $p = \prod l$ and for every $i$ such that $i \in \operatorname{dom} l$ there exists $q$ such that $l(i) = q$ and $q$ is a transposition.

$p$ is odd stands for $p$ is not even.

We now state the proposition

(29) $\mathrm{id}_{\operatorname{Seg} n}$ is even.

Let us consider $K$, $n$, and let $x$ be an element of the carrier of $K$, and let $p$ be an element of the permutations of $n$-element set. The functor $(-1)^{\operatorname{sgn}(p)}x$ yields an element of the carrier of $K$ and is defined by:

(Def.16) (i) $(-1)^{\operatorname{sgn}(p)}x = x$ if $p$ is even,

(ii) $(-1)^{\operatorname{sgn}(p)}x = -x$, otherwise.

Let $X$ be a set. Let us assume that $X$ is finite. The functor $\Omega^{\mathrm{f}}_X$ yields an element of $\operatorname{Fin} X$ and is defined as follows:

(Def.17) $\Omega^{\mathrm{f}}_X = X$.

We now state the proposition

(30) The permutations of $n$-element set is finite.

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