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Functional Sequence from a Domain to a Domain

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Summary. Definitions of functional sequences and basic operations on functional sequences from a domain to a domain, point and uniform convergent, limit of functional sequence from a domain to the set of real numbers and facts about properties of the limit of functional sequences are proved.

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The articles [11], [1], [2], [3], [13], [5], [6], [9], [8], [4], [12], [7], and [10] provide the notation and terminology for this paper. For simplicity we adopt the following rules: D, D_1, D_2 denote non-empty sets, n, k denote natural numbers, p, r denote real numbers, and f denotes a function. Let us consider D_1, D_2. A function is called a sequence of partial functions from D_1 into D_2 if:

(Def.1) $\operatorname{dom} \mathrm{it} = \mathbb{N}$ and $\operatorname{rng} \mathrm{it} \subseteq D_1 \dot{\to} D_2$.

In the sequel F, F_1, F_2 are sequences of partial functions from D_1 into D_2. Let us consider D_1, D_2, F, n. Then $F(n)$ is a partial function from D_1 to D_2.

In the sequel G, H, H_1, H_2, J are sequences of partial functions from D into $\mathbb{R}$. One can prove the following two propositions:

(1) f is a sequence of partial functions from D_1 into D_2 if and only if $\operatorname{dom} f = \mathbb{N}$ and for every n holds $f(n)$ is a partial function from D_1 to D_2.

(2) For all F_1, F_2 such that for every n holds $F_1(n) = F_2(n)$ holds $F_1 = F_2$.

The scheme *ExFuncSeq* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a partial function from $\mathcal{A}$ to $\mathcal{B}$ and states that:

there exists a sequence G of partial functions from $\mathcal{A}$ into $\mathcal{B}$ such that for every n holds $G(n) = \mathcal{F}(n)$
for all values of the parameters.

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We now define several new functors. Let us consider D, H, r. The functor $r\,H$ yields a sequence of partial functions from D into $\mathbb{R}$ and is defined as follows:

(Def.2) for every n holds $(r\,H)(n) = r\,H(n)$.

Let us consider D, H. The functor H^{-1} yielding a sequence of partial functions from D into $\mathbb{R}$ is defined by:

(Def.3) for every n holds $H^{-1}(n) = \frac{1}{H(n)}$.

The functor $-H$ yields a sequence of partial functions from D into $\mathbb{R}$ and is defined by:

(Def.4) for every n holds $(-H)(n) = -H(n)$.

The functor $|H|$ yields a sequence of partial functions from D into $\mathbb{R}$ and is defined as follows:

(Def.5) for every n holds $|H|(n) = |H(n)|$.

Let us consider D, G, H. The functor $G + H$ yields a sequence of partial functions from D into $\mathbb{R}$ and is defined by:

(Def.6) for every n holds $(G + H)(n) = G(n) + H(n)$.

The functor $G - H$ yielding a sequence of partial functions from D into $\mathbb{R}$ is defined as follows:

(Def.7) $G - H = G + -H$.

The functor $G\,H$ yields a sequence of partial functions from D into $\mathbb{R}$ and is defined as follows:

(Def.8) for every n holds $(G\,H)(n) = G(n)\,H(n)$.

Let us consider D, H, G. The functor $\frac{G}{H}$ yielding a sequence of partial functions from D into $\mathbb{R}$ is defined as follows:

(Def.9) $\frac{G}{H} = G\,H^{-1}$.

Next we state a number of propositions:

(3) $H_1 = \frac{G}{H}$ if and only if for every n holds $H_1(n) = \frac{G(n)}{H(n)}$.

(4) $H_1 = G - H$ if and only if for every n holds $H_1(n) = G(n) - H(n)$.

(5) $G + H = H + G$ and $(G + H) + J = G + (H + J)$.

(6) $G\,H = H\,G$ and $(G\,H)\,J = G\,(H\,J)$.

(7) $(G + H)\,J = G\,J + H\,J$ and $J\,(G + H) = J\,G + J\,H$.

(8) $-H = (-1)\,H$.

(9) $(G - H)\,J = G\,J - H\,J$ and $J\,G - J\,H = J\,(G - H)$.

(10) $r\,(G + H) = r\,G + r\,H$ and $r\,(G - H) = r\,G - r\,H$.

(11) $(r \cdot p)\,H = r\,(p\,H)$.

(12) $1\,H = H$.

(13) $--H = H$.

(14) $G^{-1}\,H^{-1} = (G\,H)^{-1}$.

(15) If $r \neq 0$, then $(r\,H)^{-1} = r^{-1}\,H^{-1}$.

(16) $|H|^{-1} = |H^{-1}|$.

(17) $|G\,H| = |G|\,|H|$.

(18) $|\frac{G}{H}| = \frac{|G|}{|H|}$.

(19) $|r\,H| = |r|\,|H|$.

In the sequel x is an element of D, X, Y are sets, and f is a partial function from D to $\mathbb{R}$. We now define three new constructions. Let us consider D_1, D_2, F, X. We say that X is common for elements of F if and only if:

(Def.10) $X \neq \emptyset$ and for every n holds $X \subseteq \mathrm{dom}\,F(n)$.

Let us consider D, H, x. The functor $H\#x$ yielding a sequence of real numbers is defined as follows:

(Def.11) for every n holds $(H\#x)(n) = H(n)(x)$.

Let us consider D, H, X. We say that H is point-convergent on X if and only if:

(Def.12) X is common for elements of H and there exists f such that $X = \mathrm{dom}\,f$ and for every x such that $x \in X$ and for every p such that $p > 0$ there exists k such that for every n such that $n \geq k$ holds $|H(n)(x) - f(x)| < p$.

Next we state two propositions:

(20) H is point-convergent on X if and only if X is common for elements of H and there exists f such that $X = \mathrm{dom}\,f$ and for every x such that $x \in X$ holds $H\#x$ is convergent and $\lim(H\#x) = f(x)$.

(21) H is point-convergent on X if and only if X is common for elements of H and for every x such that $x \in X$ holds $H\#x$ is convergent.

We now define two new constructions. Let us consider D, H, X. We say that H is uniform-convergent on X if and only if:

(Def.13) X is common for elements of H and there exists f such that $X = \mathrm{dom}\,f$ and for every p such that $p > 0$ there exists k such that for all n, x such that $n \geq k$ and $x \in X$ holds $|H(n)(x) - f(x)| < p$.

Let us assume that H is point-convergent on X. The functor $\lim_X H$ yielding a partial function from D to $\mathbb{R}$ is defined as follows:

(Def.14) $\mathrm{dom}\lim_X H = X$ and for every x such that $x \in \mathrm{dom}\lim_X H$ holds $(\lim_X H)(x) = \lim(H\#x)$.

We now state a number of propositions:

(22) If H is point-convergent on X, then $f = \lim_X H$ if and only if $\mathrm{dom}\,f = X$ and for every x such that $x \in X$ and for every p such that $p > 0$ there exists k such that for every n such that $n \geq k$ holds $|H(n)(x) - f(x)| < p$.

(23) If H is uniform-convergent on X, then H is point-convergent on X.

(24) If $Y \subseteq X$ and $Y \neq \emptyset$ and X is common for elements of H, then Y is common for elements of H.

(25) If $Y \subseteq X$ and $Y \neq \emptyset$ and H is point-convergent on X, then H is point-convergent on Y and $\lim_X H \restriction Y = \lim_Y H$.

(26) If $Y \subseteq X$ and $Y \neq \emptyset$ and H is uniform-convergent on X, then H is uniform-convergent on Y.

(27) If X is common for elements of H, then for every x such that $x \in X$ holds $\{x\}$ is common for elements of H.

(28) If H is point-convergent on X, then for every x such that $x \in X$ holds $\{x\}$ is common for elements of H.

(29) Suppose $\{x\}$ is common for elements of H_1 and $\{x\}$ is common for elements of H_2. Then $H_1\#x+H_2\#x = (H_1+H_2)\#x$ and $H_1\#x-H_2\#x = (H_1 - H_2)\#x$ and $(H_1\#x)\,(H_2\#x) = (H_1\,H_2)\#x$.

(30) If $\{x\}$ is common for elements of H, then $|H|\#x = |H\#x|$ and $(-H)\#x = -H\#x$.

(31) If $\{x\}$ is common for elements of H, then $(r\,H)\#x = r\,(H\#x)$.

(32) Suppose X is common for elements of H_1 and X is common for elements of H_2. Then for every x such that $x \in X$ holds $H_1\#x + H_2\#x = (H_1+H_2)\#x$ and $H_1\#x - H_2\#x = (H_1 - H_2)\#x$ and $(H_1\#x)\,(H_2\#x) = (H_1\,H_2)\#x$.

(33) If X is common for elements of H, then for every x such that $x \in X$ holds $|H|\#x = |H\#x|$ and $(-H)\#x = -H\#x$.

(34) If X is common for elements of H, then for every x such that $x \in X$ holds $(r\,H)\#x = r\,(H\#x)$.

(35) Suppose H_1 is point-convergent on X and H_2 is point-convergent on X. Then for every x such that $x \in X$ holds $H_1\#x + H_2\#x = (H_1 + H_2)\#x$ and $H_1\#x - H_2\#x = (H_1 - H_2)\#x$ and $(H_1\#x)\,(H_2\#x) = (H_1\,H_2)\#x$.

(36) If H is point-convergent on X, then for every x such that $x \in X$ holds $|H|\#x = |H\#x|$ and $(-H)\#x = -H\#x$.

(37) If H is point-convergent on X, then for every x such that $x \in X$ holds $(r\,H)\#x = r\,(H\#x)$.

(38) If X is common for elements of H_1 and X is common for elements of H_2, then X is common for elements of $H_1 + H_2$ and X is common for elements of $H_1 - H_2$ and X is common for elements of $H_1\,H_2$.

(39) If X is common for elements of H, then X is common for elements of $|H|$ and X is common for elements of $-H$.

(40) If X is common for elements of H, then X is common for elements of $r\,H$.

(41) Suppose H_1 is point-convergent on X and H_2 is point-convergent on X. Then

(i) $H_1 + H_2$ is point-convergent on X,

(ii) $\lim_X(H_1 + H_2) = \lim_X H_1 + \lim_X H_2$,

(iii) $H_1 - H_2$ is point-convergent on X,

(iv) $\lim_X(H_1 - H_2) = \lim_X H_1 - \lim_X H_2$,

(v) $H_1\,H_2$ is point-convergent on X,

(vi) $\lim_X(H_1\,H_2) = \lim_X H_1 \lim_X H_2$.

(42) If H is point-convergent on X, then $|H|$ is point-convergent on X and $\lim_X|H| = |\lim_X H|$ and $-H$ is point-convergent on X and $\lim_X(-H) =$

$-\lim_X H$.

(43) If H is point-convergent on X, then $r\,H$ is point-convergent on X and $\lim_X(r\,H) = r\lim_X H$.

(44) H is uniform-convergent on X if and only if X is common for elements of H and H is point-convergent on X and for every r such that $0 < r$ there exists k such that for all n, x such that $n \geq k$ and $x \in X$ holds $|H(n)(x) - (\lim_X H)(x)| < r$.

In the sequel H will be a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$. Let us consider n, k. Then $\max(n, k)$ is a natural number.

We now state the proposition

(45) If H is uniform-convergent on X and for every n holds $H(n)$ is continuous on X, then $\lim_X H$ is continuous on X.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[4] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[5] Jarosław Kotowicz. Partial functions from a domain to a domain. *Formalized Mathematics*, 1(**4**):697–702, 1990.

[6] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 1(**4**):703–709, 1990.

[7] Jarosław Kotowicz. Properties of real functions. *Formalized Mathematics*, 1(**4**):781–786, 1990.

[8] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[9] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[10] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[12] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[13] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

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