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# Domains of Submodules, Join and Meet of Finite Sequences of Submodules and Quotient Modules

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**Summary.** Notions of domains of submodules, join and meet of finite sequences of submodules and quotient modules. A few basic theorems and schemes related to these notions are proved.

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The papers [17], [28], [3], [4], [2], [1], [16], [5], [29], [15], [24], [20], [25], [27], [21], [18], [7], [6], [8], [26], [23], [22], [19], [14], [13], [11], [12], [9], and [10] provide the terminology and notation for this paper.

## 1. Auxiliary theorems on free-modules

For simplicity we follow a convention: $x$ is arbitrary, $K$ is an associative ring, $r$ is a scalar of $K$, $V$, $M$, $N$ are left modules over $K$, $a$, $b$, $a_1$, $a_2$ are vectors of $V$, $A$, $A_1$, $A_2$ are subsets of $V$, $l$ is a linear combination of $A$, $W$ is a submodule of $V$, and $L_1$ is a finite sequence of elements of $\mathrm{Sub}(V)$. One can prove the following propositions:

(1) If $K$ is non-trivial and $A$ is linearly independent, then $0_V \notin A$.

(2) If $a \notin A$, then $l(a) = 0_K$.

(3) If $K$ is trivial, then for every $l$ holds support $l = \emptyset$ and $\mathrm{Lin}(A)$ is trivial.

(4) If $V$ is non-trivial, then for every $A$ such that $A$ is base holds $A \neq \emptyset$.

(5) If $A_1 \cup A_2$ is linearly independent and $A_1 \cap A_2 = \emptyset$, then $\mathrm{Lin}(A_1) \cap \mathrm{Lin}(A_2) = \mathbf{0}_V$.

(6) If $A$ is base and $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$, then $V$ is the direct sum of $\mathrm{Lin}(A_1)$ and $\mathrm{Lin}(A_2)$.




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## 2. Domains of submodules

Let us consider $K$, $V$. A non-empty set is called a non empty set of submodules of $V$ if:

(Def.1) if $x \in$ it, then $x$ is a strict submodule of $V$.

Let us consider $K$, $V$. Then $\mathrm{Sub}(V)$ is a non empty set of submodules of $V$. Let us consider $K$, $V$, and let $D$ be a non empty set of submodules of $V$. We see that the element of $D$ is a strict submodule of $V$. Let us consider $K$, $V$, and let $D$ be a non empty set of submodules of $V$. One can verify that there exists a strict element of $D$.

We now state two propositions:

(7) If $x$ is an element of $\mathrm{Sub}(V)$ **qua** a non-empty set, then $x$ is an element of $\mathrm{Sub}(V)$.

(8) If $x \in \mathrm{Sub}(V)$, then $x$ is an element of $\mathrm{Sub}(V)$.

We now define two new modes. Let us consider $K$, $V$. Let us assume that $V$ is non-trivial. A strict submodule of $V$ is called a line of $V$ if:

(Def.2) there exists $a$ such that $a \neq 0_V$ and it $= \prod^* a$.

Let us consider $K$, $V$. A non-empty set is said to be a non empty set of lines of $V$ if:

(Def.3) if $x \in$ it, then $x$ is a line of $V$.

We now state two propositions:

(9) If $W$ is strict and the group structure of $W$ is strict, then $W$ is an element of $\mathrm{Sub}(V)$ **qua** a non-empty set.

(10) If $V$ is non-trivial, then every line of $V$ is an element of $\mathrm{Sub}(V)$.

We now define three new constructions. Let us consider $K$, $V$. Let us assume that $V$ is non-trivial. The functor $\mathrm{lines}(V)$ yields a non empty set of lines of $V$ and is defined as follows:

(Def.4) $\mathrm{lines}(V)$ is the set of all lines of $V$.

Let us consider $K$, $V$, and let $D$ be a non empty set of lines of $V$. We see that the element of $D$ is a line of $V$. Let us consider $K$, $V$. Let us assume that $V$ is non-trivial and $V$ is free. A strict submodule of $V$ is said to be a hiperplane of $V$ if:

(Def.5) the group structure of it is strict and there exists $a$ such that $a \neq 0_V$ and $V$ is the direct sum of $\prod^* a$ and it.

Let us consider $K$, $V$. A non-empty set is called a non empty set of hiperplanes of $V$ if:

(Def.6) if $x \in$ it, then $x$ is a hiperplane of $V$.

One can prove the following proposition

(11) If $V$ is non-trivial and $V$ is free, then every hiperplane of $V$ is an element of $\mathrm{Sub}(V)$.

Let us consider $K$, $V$. Let us assume that $V$ is non-trivial and $V$ is free. The functor hiperplanes($V$) yielding a non empty set of hiperplanes of $V$ is defined by:

(Def.7) hiperplanes($V$) is the set of all hiperplanes of $V$.

Let us consider $K$, $V$, and let $D$ be a non empty set of hiperplanes of $V$. We see that the element of $D$ is a hiperplane of $V$.

## 3. Join and meet of finite sequences of submodules

We now define two new functors. Let us consider $K$, $V$, $L_1$. The functor $\sum L_1$ yielding an element of $\mathrm{Sub}(V)$ is defined as follows:

(Def.8) $\sum L_1 = \mathrm{SubJoin}\, V \circledast L_1$.

The functor $\bigcap L_1$ yields an element of $\mathrm{Sub}(V)$ and is defined as follows:

(Def.9) $\bigcap L_1 = \mathrm{SubMeet}\, V \circledast L_1$.

The following propositions are true:

(12) For every lattice $G$ holds the join operation of $G$ is commutative and the join operation of $G$ is associative and the meet operation of $G$ is commutative and the meet operation of $G$ is associative.

(13) For every element $a$ of $\mathrm{Sub}(V)$ holds the group structure of $a$ is strict.

(14) SubJoin $V$ is commutative and SubJoin $V$ is associative and SubJoin $V$ has a unity and $\mathbf{0}_V = \mathbf{1}_{\mathrm{SubJoin}\, V}$.

(15) If the group structure of $V$ is strict, then SubMeet $V$ is commutative and SubMeet $V$ is associative and SubMeet $V$ has a unity and $\Omega_V = \mathbf{1}_{\mathrm{SubMeet}\, V}$.

## 4. Sum of subsets of module

Let us consider $K$, $V$, $A_1$, $A_2$. The functor $A_1 + A_2$ yields a subset of $V$ and is defined by:

(Def.10) $x \in A_1 + A_2$ if and only if there exist $a_1$, $a_2$ such that $a_1 \in A_1$ and $a_2 \in A_2$ and $x = a_1 + a_2$.

## 5. Vector of subset

Let us consider $K$, $V$, $A$. Let us assume that $A \neq \emptyset$. A vector of $V$ is said to be a vector of $A$ if:

(Def.11) it is an element of $A$.

One can prove the following propositions:

(16) If $A_1 \neq \emptyset$ and $A_1 \subseteq A_2$, then for every $x$ such that $x$ is a vector of $A_1$ holds $x$ is a vector of $A_2$.

(17) $a_2 \in a_1 + W$ if and only if $a_1 - a_2 \in W$.

(18) $a_1 + W = a_2 + W$ if and only if $a_1 - a_2 \in W$.

We now define two new functors. Let us consider $K$, $V$, $W$. The functor $V \looparrowleft W$ yields a non-empty set and is defined by:

(Def.12) $x \in V \looparrowleft W$ if and only if there exists $a$ such that $x = a + W$.

Let us consider $K$, $V$, $W$, $a$. The functor $a \looparrowleft W$ yields an element of $V \looparrowleft W$ and is defined as follows:

(Def.13) $a \looparrowleft W = a + W$.

We now state two propositions:

(19) For every element $x$ of $V \looparrowleft W$ there exists $a$ such that $x = a \looparrowleft W$.

(20) $a_1 \looparrowleft W = a_2 \looparrowleft W$ if and only if $a_1 - a_2 \in W$.

In the sequel $S_1$, $S_2$ will denote elements of $V \looparrowleft W$. We now define five new functors. Let us consider $K$, $V$, $W$, $S_1$. The functor $-S_1$ yields an element of $V \looparrowleft W$ and is defined by:

(Def.14) if $S_1 = a \looparrowleft W$, then $-S_1 = (-a) \looparrowleft W$.

Let us consider $S_2$. The functor $S_1 + S_2$ yields an element of $V \looparrowleft W$ and is defined by:

(Def.15) if $S_1 = a_1 \looparrowleft W$ and $S_2 = a_2 \looparrowleft W$, then $S_1 + S_2 = (a_1 + a_2) \looparrowleft W$.

Let us consider $K$, $V$, $W$. The functor $\mathrm{COMPL}(W)$ yields a unary operation on $V \looparrowleft W$ and is defined as follows:

(Def.16) $(\mathrm{COMPL}(W))(S_1) = -S_1$.

The functor $\mathrm{ADD}(W)$ yields a binary operation on $V \looparrowleft W$ and is defined by:

(Def.17) $(\mathrm{ADD}(W))(S_1, S_2) = S_1 + S_2$.

Let us consider $K$, $V$, $W$. The functor $V(W)$ yields a strict group structure and is defined by:

(Def.18) $V(W) = \langle V \looparrowleft W, \mathrm{ADD}(W), \mathrm{COMPL}(W), 0_V \looparrowleft W \rangle$.

One can prove the following proposition

(21) $a \looparrowleft W$ is an element of $V(W)$.

Let us consider $K$, $V$, $W$, $a$. The functor $a(W)$ yielding an element of $V(W)$ is defined by:

(Def.19) $a(W) = a \looparrowleft W$.

We now state four propositions:

(22) For every element $x$ of $V(W)$ there exists $a$ such that $x = a(W)$.

(23) $a_1(W) = a_2(W)$ if and only if $a_1 - a_2 \in W$.

(24) $a(W) + b(W) = (a + b)(W)$ and $-a(W) = (-a)(W)$ and $0_{V(W)} = 0_V(W)$.

(25) $V(W)$ is a strict Abelian group.

Let us consider $K$, $V$, $W$. Then $V(W)$ is a strict Abelian group.

In the sequel $S$ is an element of $V(W)$. We now define three new functors. Let us consider $K$, $V$, $W$, $r$, $S$. The functor $r \cdot S$ yielding an element of $V(W)$ is defined by:

(Def.20) if $S = a(W)$, then $r \cdot S = (r \cdot a)(W)$.

Let us consider $K$, $V$, $W$. The functor LMULT($W$) yielding a function from $[\!:$ the carrier of $K$, the carrier of $V(W)\,:\!]$ into the carrier of $V(W)$ is defined by:

(Def.21) $(\mathrm{LMULT}(W))(r, S) = r \cdot S$.

Let us consider $K$, $V$, $W$. The functor $\frac{V}{W}$ yielding a strict vector space structure over $K$ is defined as follows:

(Def.22) $\frac{V}{W} = \langle$the carrier of $V(W)$, the addition of $V(W)$, the reverse-map of $V(W)$, the zero of $V(W)$, LMULT$(W)\rangle$.

We now state two propositions:

(26) $a(W)$ is a vector of $\frac{V}{W}$.

(27) Every vector of $\frac{V}{W}$ is an element of $V(W)$.

Let us consider $K$, $V$, $W$, $a$. The functor $\frac{a}{W}$ yields a vector of $\frac{V}{W}$ and is defined as follows:

(Def.23) $\frac{a}{W} = a(W)$.

One can prove the following four propositions:

(28) For every vector $x$ of $\frac{V}{W}$ there exists $a$ such that $x = \frac{a}{W}$.

(29) $\frac{a_1}{W} = \frac{a_2}{W}$ if and only if $a_1 - a_2 \in W$.

(30) $\frac{a}{W} + \frac{b}{W} = \frac{a+b}{W}$ and $r \cdot \frac{a}{W} = \frac{r \cdot a}{W}$.

(31) $\frac{V}{W}$ is a strict left module over $K$.

Let us consider $K$, $V$, $W$. Then $\frac{V}{W}$ is a strict left module over $K$.

## 6. Quotient modules

In this article we present several logical schemes. The scheme *SetEq* deals with a unary predicate $\mathcal{P}$, and states that:

for all sets $X_1$, $X_2$ such that for an arbitrary $x$ holds $x \in X_1$ if and only if $\mathcal{P}[x]$ and for an arbitrary $x$ holds $x \in X_2$ if and only if $\mathcal{P}[x]$ holds $X_1 = X_2$

for all values of the parameter.

The scheme *DomainEq* deals with a unary predicate $\mathcal{P}$, and states that:

for all non-empty sets $X_1$, $X_2$ such that for an arbitrary $x$ holds $x \in X_1$ if and only if $\mathcal{P}[x]$ and for an arbitrary $x$ holds $x \in X_2$ if and only if $\mathcal{P}[x]$ holds $X_1 = X_2$

for all values of the parameter.

The scheme *ElementEq* concerns a set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

for all elements $X_1$, $X_2$ of $\mathcal{A}$ such that for an arbitrary $x$ holds $x \in X_1$ if and only if $\mathcal{P}[x]$ and for an arbitrary $x$ holds $x \in X_2$ if and only if $\mathcal{P}[x]$ holds $X_1 = X_2$
for all values of the parameters.

The scheme *TypeEq* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{A} = \mathcal{B}$
provided the parameters meet the following conditions:

- for an arbitrary $x$ holds $x \in \mathcal{A}$ if and only if $\mathcal{P}[x]$,
- for an arbitrary $x$ holds $x \in \mathcal{B}$ if and only if $\mathcal{P}[x]$.

The scheme *FuncEq* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ and states that:

for all functions $f_1$, $f_2$ from $\mathcal{A}$ into $\mathcal{B}$ such that for every element $x$ of $\mathcal{A}$ holds $f_1(x) = \mathcal{F}(x)$ and for every element $x$ of $\mathcal{A}$ holds $f_2(x) = \mathcal{F}(x)$ holds $f_1 = f_2$
for all values of the parameters.

The scheme *UnOpEq* deals with a non-empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ and states that:

for all unary operations $f_1$, $f_2$ on $\mathcal{A}$ such that for every element $a$ of $\mathcal{A}$ holds $f_1(a) = \mathcal{F}(a)$ and for every element $a$ of $\mathcal{A}$ holds $f_2(a) = \mathcal{F}(a)$ holds $f_1 = f_2$
for all values of the parameters.

The scheme *BinOpEq* concerns a non-empty set $\mathcal{A}$ and a binary functor $\mathcal{F}$ and states that:

for all binary operations $f_1$, $f_2$ on $\mathcal{A}$ such that for all elements $a$, $b$ of $\mathcal{A}$ holds $f_1(a, b) = \mathcal{F}(a,b)$ and for all elements $a$, $b$ of $\mathcal{A}$ holds $f_2(a, b) = \mathcal{F}(a,b)$ holds $f_1 = f_2$
for all values of the parameters.

The scheme *TriOpEq* deals with a non-empty set $\mathcal{A}$ and a ternary functor $\mathcal{F}$ and states that:

for all ternary operations $f_1$, $f_2$ on $\mathcal{A}$ such that for all elements $a$, $b$, $c$ of $\mathcal{A}$ holds $f_1(a, b, c) = \mathcal{F}(a,b,c)$ and for all elements $a$, $b$, $c$ of $\mathcal{A}$ holds $f_2(a, b, c) = \mathcal{F}(a,b,c)$ holds $f_1 = f_2$
for all values of the parameters.

The scheme *QuaOpEq* deals with a non-empty set $\mathcal{A}$ and a 4-ary functor $\mathcal{F}$ and states that:

for all quadrary operations $f_1$, $f_2$ on $\mathcal{A}$ such that for all elements $a$, $b$, $c$, $d$ of $\mathcal{A}$ holds $f_1(a, b, c, d) = \mathcal{F}(a,b,c,d)$ and for all elements $a$, $b$, $c$, $d$ of $\mathcal{A}$ holds $f_2(a, b, c, d) = \mathcal{F}(a,b,c,d)$ holds $f_1 = f_2$
for all values of the parameters.

The scheme *Fraenkel1_Ex* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a subset $S$ of $\mathcal{B}$ such that $S = \{\mathcal{F}(x) : \mathcal{P}[x]\}$, where $x$ ranges over elements of $\mathcal{A}$
for all values of the parameters.

The scheme *Fr_0* concerns a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the parameters meet the following requirement:

- $\mathcal{B} \in \{a : \mathcal{P}[a]\}$, where $a$ ranges over elements of $\mathcal{A}$.

The scheme *Fr_1* deals with a set $\mathcal{A}$, a non-empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{C} \in \mathcal{A}$ if and only if $\mathcal{P}[\mathcal{C}]$

provided the following condition is satisfied:

- $\mathcal{A} = \{a : \mathcal{P}[a]\}$, where $a$ ranges over elements of $\mathcal{B}$.

The scheme *Fr_2* concerns a set $\mathcal{A}$, a non-empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{C}]$

provided the following conditions are met:

- $\mathcal{C} \in \mathcal{A}$,
- $\mathcal{A} = \{a : \mathcal{P}[a]\}$, where $a$ ranges over elements of $\mathcal{B}$.

The scheme *Fr_3* concerns a constant $\mathcal{A}$, a set $\mathcal{B}$, a non-empty set $\mathcal{C}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{A} \in \mathcal{B}$ if and only if there exists an element $a$ of $\mathcal{C}$ such that $\mathcal{A} = a$ and $\mathcal{P}[a]$

provided the parameters meet the following condition:

- $\mathcal{B} = \{a : \mathcal{P}[a]\}$, where $a$ ranges over elements of $\mathcal{C}$.

The scheme *Fr_4* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a set $\mathcal{C}$, an element $\mathcal{D}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$, and two binary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

$\mathcal{D} \in \mathcal{F}(\mathcal{C})$ if and only if for every element $b$ of $\mathcal{B}$ such that $b \in \mathcal{C}$ holds $\mathcal{P}[\mathcal{D}, b]$

provided the parameters meet the following conditions:

- $\mathcal{F}(\mathcal{C}) = \{a : \mathcal{Q}[a, \mathcal{C}]\}$, where $a$ ranges over elements of $\mathcal{A}$,
- $\mathcal{Q}[\mathcal{D}, \mathcal{C}]$ if and only if for every element $b$ of $\mathcal{B}$ such that $b \in \mathcal{C}$ holds $\mathcal{P}[\mathcal{D}, b]$.

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