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The Product and the Determinant of Matrices with Entries in a Field

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Summary. Concerned with a generalization of concepts introduced in [17], i.e. there are introduced the sum and the product of matrices of any dimension of elements of any field.

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The articles [15], [28], [10], [11], [5], [7], [6], [12], [16], [20], [27], [19], [23], [13], [9], [8], [21], [26], [1], [17], [25], [18], [4], [3], [24], [29], [2], [22], and [14] provide the notation and terminology for this paper.

For simplicity we follow a convention: i, j, k, l, n, m denote natural numbers, I, J, D denote non empty sets, K denotes a field, a denotes an element of D, and p, q denote finite sequences of elements of D.

We now state two propositions:

(1) If $n = n + k$, then $k = 0$.

(2) For every natural number n holds $n = 0$ or $n = 1$ or $n = 2$ or $n > 2$.

In the sequel A, B will denote matrices over K of dimension $n \times m$.

Let us consider K, n, m. The functor $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n \times m}$ yields a matrix over K of dimension $n \times m$ and is defined as follows:

$$\text{(Def.1)} \quad \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n \times m} = n \longmapsto (m \longmapsto 0_K).$$

Let us consider K and let A be a matrix over K. The functor $-A$ yields a matrix over K and is defined by:

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(Def.2) $\operatorname{len}(-A) = \operatorname{len} A$ and $\operatorname{width}(-A) = \operatorname{width} A$ and for all i, j such that $\langle i, j\rangle \in$ the indices of A holds $(-A)_{i,j} = -A_{i,j}$.

Let us consider K and let A, B be matrices over K. Let us assume that $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$. The functor $A+B$ yielding a matrix over K is defined as follows:

(Def.3) $\operatorname{len}(A+B) = \operatorname{len} A$ and $\operatorname{width}(A+B) = \operatorname{width} A$ and for all i, j such that $\langle i, j\rangle \in$ the indices of A holds $(A+B)_{i,j} = A_{i,j} + B_{i,j}$.

The following proposition is true

(3) For all i, j such that $\langle i, j\rangle \in$ the indices of $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n\times m}$ holds $\left(\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n\times m}\right)_{i,j} = 0_K$.

In the sequel A, B denote matrices over K.

The following propositions are true:

(4) For all matrices A, B over K such that $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ holds $A+B = B+A$.

(5) For all matrices A, B, C over K such that $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{len} A = \operatorname{len} C$ and $\operatorname{width} A = \operatorname{width} B$ and $\operatorname{width} A = \operatorname{width} C$ holds $(A+B)+C = A+(B+C)$.

(6) For every matrix A over K of dimension $n \times m$ holds $A + \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n\times m} = A$.

(7) For every matrix A over K of dimension $n \times m$ holds $A + -A = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n\times m}$

Let us consider K and let A, B be matrices over K. Let us assume that $\operatorname{width} A = \operatorname{len} B$. The functor $A \cdot B$ yields a matrix over K and is defined as follows:

(Def.4) $\operatorname{len}(A \cdot B) = \operatorname{len} A$ and $\operatorname{width}(A \cdot B) = \operatorname{width} B$ and for all i, j such that $\langle i, j\rangle \in$ the indices of $A \cdot B$ holds $(A \cdot B)_{i,j} = \operatorname{Line}(A, i) \cdot B_{\square,j}$.

Let us consider n, k, m, let us consider K, let A be a matrix over K of dimension $n \times k$, and let B be a matrix over K of dimension $\operatorname{width} A \times m$. Then $A \cdot B$ is a matrix over K of dimension $\operatorname{len} A \times \operatorname{width} B$.

Let us consider K, let M be a matrix over K, and let a be an element of the carrier of K. The functor $a \cdot M$ yields a matrix over K and is defined by:

(Def.5) $\mathrm{len}(a \cdot M) = \mathrm{len}\, M$ and $\mathrm{width}(a \cdot M) = \mathrm{width}\, M$ and for all i, j such that $\langle i, j \rangle \in$ the indices of M holds $(a \cdot M)_{i,j} = a \cdot M_{i,j}$.

Let us consider K, let M be a matrix over K, and let a be an element of the carrier of K. The functor $M \cdot a$ yields a matrix over K and is defined by:

(Def.6) $M \cdot a = a \cdot M$.

One can prove the following propositions:

(8) For all finite sequences p, q of elements of the carrier of K such that $\mathrm{len}\, p = \mathrm{len}\, q$ holds $\mathrm{len}(p \bullet q) = \mathrm{len}\, p$ and $\mathrm{len}(p \bullet q) = \mathrm{len}\, q$.

(9) For all i, l such that $\langle i, l \rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ and $l = i$ holds $\mathrm{Line}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, i)(l) = 1_K$.

(10) For all i, l such that $\langle i, l \rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ and $l \neq i$ holds $\mathrm{Line}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, i)(l) = 0_K$.

(11) For all l, j such that $\langle l, j \rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ and $l = j$ holds $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})_{\square, j}(l) = 1_K$.

(12) For all l, j such that $\langle l, j \rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ and $l \neq j$ holds $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})_{\square, j}(l) = 0_K$.

(13) $\sum(n \longmapsto 0_K) = 0_K$.

(14) Let p be a finite sequence of elements of the carrier of K and given i. Suppose $i \in \mathrm{Seg}\,\mathrm{len}\, p$ and for every k such that $k \in \mathrm{Seg}\,\mathrm{len}\, p$ and $k \neq i$ holds $p(k) = 0_K$. Then $\sum p = p(i)$.

(15) For all finite sequences p, q of elements of the carrier of K holds $\mathrm{len}(p \bullet$

$q) = \min(\operatorname{len} p, \operatorname{len} q)$.

(16) Let p, q be finite sequences of elements of the carrier of K and given i. Suppose $i \in \operatorname{Seg} \operatorname{len} p$ and $p(i) = 1_K$ and for every k such that $k \in \operatorname{Seg} \operatorname{len} p$ and $k \neq i$ holds $p(k) = 0_K$. Given j. Suppose $j \in \operatorname{Seg} \operatorname{len}(p \bullet q)$. Then if $i = j$, then $(p \bullet q)(j) = q(i)$ and if $i \neq j$, then $(p \bullet q)(j) = 0_K$.

(17) For all i, j such that $\langle i, j\rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ holds if $i = j$, then $\operatorname{Line}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, i)(j) = 1_K$ and if $i \neq j$, then $\operatorname{Line}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, i)(j) = 0_K$.

(18) For all i, j such that $\langle i, j\rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ holds if $i = j$, then $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})_{\square, j}(i) = 1_K$ and if $i \neq j$, then $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})_{\square, j}(i) = 0_K$.

(19) Let p, q be finite sequences of elements of the carrier of K and given i. Suppose $i \in \operatorname{Seg} \operatorname{len} p$ and $i \in \operatorname{Seg} \operatorname{len} q$ and $p(i) = 1_K$ and for every k such that $k \in \operatorname{Seg} \operatorname{len} p$ and $k \neq i$ holds $p(k) = 0_K$. Then $\sum(p \bullet q) = q(i)$.

(20) For every matrix A over K of dimension n holds $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n} \cdot A = A$.

(21) For every matrix A over K of dimension n holds $A \cdot \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n} = A$.

(22) For all elements a, b of the carrier of K holds $\langle\langle a\rangle\rangle \cdot \langle\langle b\rangle\rangle = \langle\langle a \cdot b\rangle\rangle$.

(23) For all elements a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 of the carrier of K holds $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 \cdot a_2 + b_1 \cdot c_2 & a_1 \cdot b_2 + b_1 \cdot d_2 \\ c_1 \cdot a_2 + d_1 \cdot c_2 & c_1 \cdot b_2 + d_1 \cdot d_2 \end{pmatrix}$.

(24) For all matrices A, B over K such that width $A = \operatorname{len} B$ and width $B \neq 0$ holds $(A \cdot B)^{\mathrm{T}} = B^{\mathrm{T}} \cdot A^{\mathrm{T}}$.

Let I, J be non empty sets, let X be an element of $\operatorname{Fin} I$, and let Y be an element of $\operatorname{Fin} J$. Then $[: X, Y :]$ is an element of $\operatorname{Fin}[: I, J :]$.

Let I, J, D be non empty sets, let G be a binary operation on D, let f be a function from I into D, and let g be a function from J into D. Then $G \circ (f, g)$ is a function from $[: I, J :]$ into D.

The following propositions are true:

(25) Let I, J, D be non empty sets, and let F, G be binary operations on D, and let f be a function from I into D, and let g be a function from J into D, and let X be an element of $\operatorname{Fin} I$, and let Y be an element of $\operatorname{Fin} J$. Suppose F is commutative and associative but $[: Y, X :] \neq \emptyset$ or F has a unity but G is commutative. Then $F\text{-}\sum_{[: X, Y :]}(G \circ (f, g)) = F\text{-}\sum_{[: Y, X :]}(G \circ (g, f))$.

(26) Let I, J be non empty sets, and let F, G be binary operations on D, and let f be a function from I into D, and let g be a function from J into D. Suppose F is commutative and associative and has a unity. Let x be an element of I and let y be an element of J. Then $F\text{-}\sum_{[: \{x\}, \{y\} :]}(G \circ (f, g)) = F\text{-}\sum_{\{x\}} G^{\circ}(f, F\text{-}\sum_{\{y\}} g)$.

(27) Let I, J be non empty sets, and let F, G be binary operations on D, and let f be a function from I into D, and let g be a function from J into D, and let X be an element of $\operatorname{Fin} I$, and let Y be an element of $\operatorname{Fin} J$. Suppose F is commutative and associative and has a unity and an inverse operation and G is distributive w.r.t. F. Let x be an element of I. Then $F\text{-}\sum_{[: \{x\}, Y :]}(G \circ (f, g)) = F\text{-}\sum_{\{x\}} G^{\circ}(f, F\text{-}\sum_{Y} g)$.

(28) Let I, J be non empty sets, and let F, G be binary operations on D, and let f be a function from I into D, and let g be a function from J into D, and let X be an element of $\operatorname{Fin} I$, and let Y be an element of $\operatorname{Fin} J$. Suppose F is commutative and associative and has a unity and an inverse operation and G is distributive w.r.t. F. Then $F\text{-}\sum_{[: X, Y :]}(G \circ (f, g)) = F\text{-}\sum_{X} G^{\circ}(f, F\text{-}\sum_{Y} g)$.

(29) Let I, J be non empty sets, and let F, G be binary operations on D, and let f be a function from I into D, and let g be a function from J into D. Suppose F is commutative and associative and has a unity and G is commutative. Let x be an element of I and let y be an element of J. Then $F\text{-}\sum_{[: \{x\}, \{y\} :]}(G \circ (f, g)) = F\text{-}\sum_{\{y\}} G^{\circ}(F\text{-}\sum_{\{x\}} f, g)$.

(30) Let I, J be non empty sets, and let F, G be binary operations on D, and let f be a function from I into D, and let g be a function from J into D, and let X be an element of $\operatorname{Fin} I$, and let Y be an element of $\operatorname{Fin} J$. Suppose that

(i) F is commutative and associative and has a unity and an inverse operation, and

(ii) G is distributive w.r.t. F and commutative.

Then $F\text{-}\sum_{[: X, Y :]}(G \circ (f, g)) = F\text{-}\sum_{Y} G^{\circ}(F\text{-}\sum_{X} f, g)$.

(31) Let I, J be non empty sets, and let F be a binary operation on D, and let f be a function from $[: I, J :]$ into D, and let g be a function from I into D, and let Y be an element of Fin J. Suppose F is commutative and associative and has a unity and an inverse operation. Let x be an element of I. If for every element i of I holds $g(i) = F\text{-}\sum_Y(\text{curry } f)(i)$, then $F\text{-}\sum_{[: \{x\}, Y :]} f = F\text{-}\sum_{\{x\}} g$.

(32) Let I, J be non empty sets, and let F be a binary operation on D, and let f be a function from $[: I, J :]$ into D, and let g be a function from I into D, and let X be an element of Fin I, and let Y be an element of Fin J. Suppose for every element i of I holds $g(i) = F\text{-}\sum_Y(\text{curry } f)(i)$ and F is commutative and associative and has a unity and an inverse operation. Then $F\text{-}\sum_{[: X, Y :]} f = F\text{-}\sum_X g$.

(33) Let I, J be non empty sets, and let F be a binary operation on D, and let f be a function from $[: I, J :]$ into D, and let g be a function from J into D, and let X be an element of Fin I. Suppose F is commutative and associative and has a unity and an inverse operation. Let y be an element of J. If for every element j of J holds $g(j) = F\text{-}\sum_X(\text{curry}' f)(j)$, then $F\text{-}\sum_{[: X, \{y\} :]} f = F\text{-}\sum_{\{y\}} g$.

(34) Let I, J be non empty sets, and let F be a binary operation on D, and let f be a function from $[: I, J :]$ into D, and let g be a function from J into D, and let X be an element of Fin I, and let Y be an element of Fin J. Suppose for every element j of J holds $g(j) = F\text{-}\sum_X(\text{curry}' f)(j)$ and F is commutative and associative and has a unity and an inverse operation. Then $F\text{-}\sum_{[: X, Y :]} f = F\text{-}\sum_Y g$.

(35) For all matrices A, B, C over K such that width A = len B and width B = len C holds $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

In the sequel p will be an element of the permutations of n-element set.

Let us consider n, K, let M be a matrix over K of dimension n, and let p be an element of the permutations of n-element set. The functor p-Path M yields a finite sequence of elements of the carrier of K and is defined as follows:

(Def.7) len(p-Path M) = n and for all i, j such that $i \in$ dom(p-Path M) and $j = p(i)$ holds (p-Path M)(i) = $M_{i,j}$.

Let us consider n, K and let M be a matrix over K of dimension n. The product on paths of M yields a function from the permutations of n-element set into the carrier of K and is defined by the condition (Def.8).

(Def.8) Let p be an element of the permutations of n-element set. Then (the product on paths of M)(p) = $(-1)^{\text{sgn}(p)}$(the multiplication of $K \circledast$ (p-Path M)).

Let us consider n, let us consider K, and let M be a matrix over K of dimension n. The functor Det M yields an element of the carrier of K and is defined as follows:

(Def.9) Det M = (the addition of K)-$\sum_{\Omega^{\text{f}}_{\text{the permutations of } n\text{-element set}}}$(the product on paths of M).

In the sequel a will be an element of the carrier of K.

The following proposition is true

(36) $\mathrm{Det}\langle\langle a\rangle\rangle = a$.

Let us consider n, let us consider K, and let M be a matrix over K of dimension n. The diagonal of M yields a finite sequence of elements of the carrier of K and is defined as follows:

(Def.10) len (the diagonal of M) $= n$ and for every i such that $i \in \mathrm{Seg}\, n$ holds (the diagonal of $M)(i) = M_{i,i}$.

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Introduction to Theory of Rearrangement [1]

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Summary. An introduction to the rearrangement theory for finite functions (e.g. with the finite domain and codomain). The notion of generators and cogenerators of finite sets (equivalent to the order in the language of finite sequences) has been defined. The notion of rearrangement for a function into finite set is presented. Some basic properties of these notions have been proved.

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The terminology and notation used here are introduced in the following articles: [15], [5], [3], [1], [8], [10], [2], [16], [6], [4], [7], [12], [13], [9], [11], and [14].

Let D be a non empty set, let F be a partial function from D to $\mathbb{R}$, and let r be a real number. Then $r\, F$ is an element of $D \dot{\rightarrow} \mathbb{R}$.

A finite sequence has cardinality by index if:

(Def.1) For every n such that $1 \leq n$ and $n \leq \operatorname{len} \mathrm{it}$ holds $\operatorname{card} \mathrm{it}(n) = n$.

A finite sequence is ascending if:

(Def.2) For every n such that $1 \leq n$ and $n \leq \operatorname{len} \mathrm{it} - 1$ holds $\mathrm{it}(n) \subseteq \mathrm{it}(n+1)$.

Let X be a set. A finite sequence of elements of X has length by cardinality if:

(Def.3) $\operatorname{len} \mathrm{it} = \operatorname{card} \bigcup X$.

Let D be a non empty finite set. Note that there exists a finite sequence of elements of 2^D which is ascending and has cardinality by index and length by cardinality.

Let D be a non empty finite set. A rearrangement generator of D is an ascending finite sequence of elements of 2^D with cardinality by index and length by cardinality.

One can prove the following propositions:

[1]Dedicated to Professor Tsuyoshi Ando on his sixtieth birthday.

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(1) For every finite sequence a of elements of 2^D holds a has length by cardinality iff $\operatorname{len} a = \operatorname{card} D$.

(2) Let a be a finite sequence. Then a is ascending if and only if for all n, m such that $n \leq m$ and $n \in \operatorname{dom} a$ and $m \in \operatorname{dom} a$ holds $a(n) \subseteq a(m)$.

(3) For every finite sequence a of elements of 2^D with cardinality by index and length by cardinality holds $a(\operatorname{len} a) = D$.

(4) For every finite sequence a of elements of 2^D with length by cardinality holds $\operatorname{len} a \neq 0$.

(5) Let a be an ascending finite sequence of elements of 2^D with cardinality by index and given n, m. If $n \in \operatorname{dom} a$ and $m \in \operatorname{dom} a$ and $n \neq m$, then $a(n) \neq a(m)$.

(6) Let a be an ascending finite sequence of elements of 2^D with cardinality by index and given n. If $1 \leq n$ and $n \leq \operatorname{len} a - 1$, then $a(n) \neq a(n+1)$.

(7) For every finite sequence a of elements of 2^D with cardinality by index such that $n \in \operatorname{dom} a$ holds $a(n) \neq \emptyset$.

(8) Let a be a finite sequence of elements of 2^D with cardinality by index. If $1 \leq n$ and $n \leq \operatorname{len} a - 1$, then $a(n+1) \setminus a(n) \neq \emptyset$.

(9) Let a be a finite sequence of elements of 2^D with cardinality by index and length by cardinality. Then there exists an element d of D such that $a(1) = \{d\}$.

(10) Let a be an ascending finite sequence of elements of 2^D with cardinality by index. Suppose $1 \leq n$ and $n \leq \operatorname{len} a - 1$. Then there exists an element d of D such that $a(n+1) \setminus a(n) = \{d\}$ and $a(n+1) = a(n) \cup \{d\}$ and $a(n+1) \setminus \{d\} = a(n)$.

Let D be a non empty finite set and let A be a rearrangement generator of D. The functor co-Gen(A) yielding a rearrangement generator of D is defined by:

(Def.4) For every m such that $1 \leq m$ and $m \leq \operatorname{len}$ co-Gen$(A) - 1$ holds $(\text{co-Gen}(A))(m) = D \setminus A(\operatorname{len} A - m)$.

One can prove the following two propositions:

(11) For every rearrangement generator A of D holds co-Gen(co-Gen(A)) = A.

(12) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{len} \mathrm{MIM}(\mathrm{FinS}(F, D)) = \operatorname{len} \mathrm{CHI}(A, C)$.

Let D, C be non empty finite set, let A be a rearrangement generator of C, and let F be a partial function from D to $\mathbb{R}$. The functor $F_A^\wedge$ yields a partial function from C to $\mathbb{R}$ and is defined by:

(Def.5) $F_A^\wedge = \sum(\mathrm{MIM}(\mathrm{FinS}(F, D))\, \mathrm{CHI}(A, C))$.

The functor $F_A^\vee$ yields a partial function from C to $\mathbb{R}$ and is defined as follows:

(Def.6) $F_A^\vee = \sum(\mathrm{MIM}(\mathrm{FinS}(F, D))\, \mathrm{CHI}(\text{co-Gen}(A), C))$.

Next we state a number of propositions:

(13) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{dom} F_A^\wedge = C$.

(14) Let c be an element of C, and let F be a partial function from D to $\mathbb{R}$, and let A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then

(i) if $c \in A(1)$, then $(\mathrm{MIM}(\mathrm{FinS}(F, D))\,\mathrm{CHI}(A, C))\#c = \mathrm{MIM}(\mathrm{FinS}(F, D))$, and

(ii) for every n such that $1 \leq n$ and $n < \operatorname{len} A$ and $c \in A(n+1) \setminus A(n)$ holds $(\mathrm{MIM}(\mathrm{FinS}(F, D))\,\mathrm{CHI}(A, C))\#c = (n \longmapsto (0\ \mathbf{qua}\ \text{real number}))$ $^\frown\mathrm{MIM}((\mathrm{FinS}(F, D))_{\restriction n})$.

(15) Let c be an element of C, and let F be a partial function from D to $\mathbb{R}$, and let A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then if $c \in A(1)$, then $(F_A^\wedge)(c) = (\mathrm{FinS}(F, D))(1)$ and for every n such that $1 \leq n$ and $n < \operatorname{len} A$ and $c \in A(n+1) \setminus A(n)$ holds $(F_A^\wedge)(c) = (\mathrm{FinS}(F, D))(n+1)$.

(16) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{rng} F_A^\wedge = \operatorname{rng} \mathrm{FinS}(F, D)$.

(17) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then $F_A^\wedge$ and $\mathrm{FinS}(F, D)$ are fiberwise equipotent.

(18) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\mathrm{FinS}(F_A^\wedge, C) = \mathrm{FinS}(F, D)$.

(19) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\sum_{\kappa=0}^{C} F_A^\wedge(\kappa) = \sum_{\kappa=0}^{D} F(\kappa)$.

(20) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\mathrm{FinS}((F_A^\wedge) - r, C) = \mathrm{FinS}(F - r, D)$ and $\sum_{\kappa=0}^{C}((F_A^\wedge) - r)(\kappa) = \sum_{\kappa=0}^{D}(F - r)(\kappa)$.

(21) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{dom} F_A^\vee = C$.

(22) Let c be an element of C, and let F be a partial function from D to $\mathbb{R}$, and let A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then if $c \in (\text{co-Gen}(A))(1)$, then $(F_A^\vee)(c) = (\mathrm{FinS}(F, D))(1)$ and for every n such that $1 \leq n$ and $n < \operatorname{len} \text{co-Gen}(A)$ and $c \in (\text{co-Gen}(A))(n+1) \setminus (\text{co-Gen}(A))(n)$ holds $(F_A^\vee)(c) = (\mathrm{FinS}(F, D))(n+1)$.

(23) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{rng} F_A^\vee = \operatorname{rng} \mathrm{FinS}(F, D)$.

(24) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then $F_A^\vee$ and

FinS(F, D) are fiberwise equipotent.

(25) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{FinS}(F_A^{\vee}, C) = \operatorname{FinS}(F, D)$.

(26) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\sum_{\kappa=0}^{C} F_A^{\vee}(\kappa) = \sum_{\kappa=0}^{D} F(\kappa)$.

(27) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{FinS}((F_A^{\vee}) - r, C) = \operatorname{FinS}(F - r, D)$ and $\sum_{\kappa=0}^{C}((F_A^{\vee}) - r)(\kappa) = \sum_{\kappa=0}^{D}(F - r)(\kappa)$.

(28) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then $F_A^{\vee}$ and $F_A^{\wedge}$ are fiberwise equipotent and $\operatorname{FinS}(F_A^{\vee}, C) = \operatorname{FinS}(F_A^{\wedge}, C)$ and $\sum_{\kappa=0}^{C} F_A^{\vee}(\kappa) = \sum_{\kappa=0}^{C} F_A^{\wedge}(\kappa)$.

(29) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then $\max_{+}((F_A^{\wedge}) - r)$ and $\max_{+}(F - r)$ are fiberwise equipotent and $\operatorname{FinS}(\max_{+}((F_A^{\wedge}) - r), C) = \operatorname{FinS}(\max_{+}(F - r), D)$ and $\sum_{\kappa=0}^{C} \max_{+}((F_A^{\wedge}) - r)(\kappa) = \sum_{\kappa=0}^{D} \max_{+}(F - r)(\kappa)$.

(30) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then $\max_{-}((F_A^{\wedge}) - r)$ and $\max_{-}(F - r)$ are fiberwise equipotent and $\operatorname{FinS}(\max_{-}((F_A^{\wedge}) - r), C) = \operatorname{FinS}(\max_{-}(F - r), D)$ and $\sum_{\kappa=0}^{C} \max_{-}((F_A^{\wedge}) - r)(\kappa) = \sum_{\kappa=0}^{D} \max_{-}(F - r)(\kappa)$.

(31) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} D = \operatorname{card} C$, then $\operatorname{len} \operatorname{FinS}(F_A^{\wedge}, C) = \operatorname{card} C$ and $1 \leq \operatorname{len} \operatorname{FinS}(F_A^{\wedge}, C)$.

(32) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} D = \operatorname{card} C$ and $n \in \operatorname{dom} A$, then $\operatorname{FinS}(F_A^{\wedge}, C) \restriction n = \operatorname{FinS}(F_A^{\wedge}, A(n))$.

(33) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} D = \operatorname{card} C$, then $(F - r)_A^{\wedge} = (F_A^{\wedge}) - r$.

(34) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then $\max_{+}((F_A^{\vee}) - r)$ and $\max_{+}(F - r)$ are fiberwise equipotent and $\operatorname{FinS}(\max_{+}((F_A^{\vee}) - r), C) = \operatorname{FinS}(\max_{+}(F - r), D)$ and $\sum_{\kappa=0}^{C} \max_{+}((F_A^{\vee}) - r)(\kappa) = \sum_{\kappa=0}^{D} \max_{+}(F - r)(\kappa)$.

(35) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then $\max_{-}((F_A^{\vee}) - r)$ and $\max_{-}(F - r)$ are fiberwise equipotent and $\operatorname{FinS}(\max_{-}((F_A^{\vee}) - r), C) = \operatorname{FinS}(\max_{-}(F - r), D)$ and $\sum_{\kappa=0}^{C} \max_{-}((F_A^{\vee}) - r)(\kappa) = \sum_{\kappa=0}^{D} \max_{-}(F - r)(\kappa)$.

(36) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} D = \operatorname{card} C$, then $\operatorname{len} \operatorname{FinS}(F_A^\vee, C) = \operatorname{card} C$ and $1 \leq \operatorname{len} \operatorname{FinS}(F_A^\vee, C)$.

(37) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} D = \operatorname{card} C$ and $n \in \operatorname{dom} A$, then $\operatorname{FinS}(F_A^\vee, C) \restriction n = \operatorname{FinS}(F_A^\vee, (\text{co-Gen}(A))(n))$.

(38) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. If F is total and $\operatorname{card} D = \operatorname{card} C$, then $(F - r)_A^\vee = (F_A^\vee) - r$.

(39) Let F be a partial function from D to $\mathbb{R}$ and let A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} D = \operatorname{card} C$. Then $F_A^\wedge$ and F are fiberwise equipotent and $F_A^\vee$ and F are fiberwise equipotent and $\operatorname{rng} F_A^\wedge = \operatorname{rng} F$ and $\operatorname{rng} F_A^\vee = \operatorname{rng} F$.

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Many-sorted Sets

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Summary. The article deals with parameterized families of sets. When treated in a similar way as sets (due to systematic overloading notation used for sets) they are called many sorted sets. For instance, if x and X are two many-sorted sets (with the same set of indices I) then relation $x \in X$ is defined as $\forall_{i \in I} x_i \in X_i$.

I was prompted by a remark in a paper by Tarlecki and Wirsing: "Throughout the paper we deal with many-sorted sets, functions, relations etc. ... We feel free to use any standard set-theoretic notation without explicit use of indices" [3, p.97]. The aim of this work was to check the feasibility of such approach in Mizar. It works.

Let us observe some peculiarities:

- empty set (i.e. the many sorted set with empty set of indices) belongs to itself (theorem 133),
- we get two different inclusions $X \subseteq Y$ iff $\forall_{i \in I} X_i \subseteq Y_i$ and $X \sqsubseteq Y$ iff $\forall_x x \in X \Rightarrow x \in Y$ equivalent only for sets that yield non empty values.

Therefore the care is advised.

MML Identifier: PBOOLE.

The articles [5], [1], [4], and [2] provide the terminology and notation for this paper.

1. Preliminaries

In the sequel i, e will be arbitrary.

A function is empty yielding if:

(Def.1) For every i such that $i \in \operatorname{dom} \mathrm{it}$ holds $\mathrm{it}(i)$ is empty.

A function is non empty set yielding if:

(Def.2) For every i such that $i \in \operatorname{dom} \mathrm{it}$ holds $\mathrm{it}(i)$ is non empty.

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Next we state two propositions:

(1) For every function f such that f is non empty yielding holds rng f has non empty elements.

(2) For every function f holds f is empty yielding iff $f = \emptyset$ or rng $f = \{\emptyset\}$.

In the sequel I denotes a set.

Let us consider I. A function is said to be a many sorted set of I if:

(Def.3) $\operatorname{dom} \mathrm{it} = I$.

In the sequel x, y, z, X, Y, Z, V are many sorted sets of I.

The scheme *Kuratowski Function* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

> There exists a many sorted set f of $\mathcal{A}$ such that for every e such that $e \in \mathcal{A}$ holds $f(e) \in \mathcal{F}(e)$

provided the following requirement is met:

- For every e such that $e \in \mathcal{A}$ holds $\mathcal{F}(e) \neq \emptyset$.

Let us consider I, X, Y. The predicate $X \in Y$ is defined by:

(Def.4) For every i such that $i \in I$ holds $X(i) \in Y(i)$.

The predicate $X \subseteq Y$ is defined by:

(Def.5) For every i such that $i \in I$ holds $X(i) \subseteq Y(i)$.

The scheme *PSeparation* deals with a set $\mathcal{A}$, a many sorted set $\mathcal{B}$ of $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

> There exists a many sorted set X of $\mathcal{A}$ such that for every set i holds if $i \in \mathcal{A}$, then for every e holds $e \in X(i)$ iff $e \in \mathcal{B}(i)$ and $\mathcal{P}[i, e]$

for all values of the parameters.

One can prove the following proposition

(3) If for every i such that $i \in I$ holds $X(i) = Y(i)$, then $X = Y$.

Let us consider I. The functor $\emptyset_I$ yields a many sorted set of I and is defined by:

(Def.6) $\emptyset_I = I \longmapsto \emptyset$.

Let us consider X, Y. The functor $X \cup Y$ yielding a many sorted set of I is defined by:

(Def.7) For every i such that $i \in I$ holds $(X \cup Y)(i) = X(i) \cup Y(i)$.

The functor $X \cap Y$ yielding a many sorted set of I is defined by:

(Def.8) For every i such that $i \in I$ holds $(X \cap Y)(i) = X(i) \cap Y(i)$.

The functor $X \setminus Y$ yields a many sorted set of I and is defined as follows:

(Def.9) For every i such that $i \in I$ holds $(X \setminus Y)(i) = X(i) \setminus Y(i)$.

We say that X overlaps Y if and only if:

(Def.10) For every i such that $i \in I$ holds $X(i)$ meets $Y(i)$.

We say that X misses Y if and only if:

(Def.11) For every i such that $i \in I$ holds $X(i)$ misses $Y(i)$.

Let us consider I, X, Y. The functor $X \dot{-} Y$ yielding a many sorted set of I is defined as follows:

(Def.12) $X \dot{-} Y = (X \setminus Y) \cup (Y \setminus X)$.

Next we state several propositions:

(4) For every i such that $i \in I$ holds $(X \dot{-} Y)(i) = X(i) \dot{-} Y(i)$.

(5) For every i such that $i \in I$ holds $\emptyset_I(i) = \emptyset$.

(6) If for every i such that $i \in I$ holds $X(i) = \emptyset$, then $X = \emptyset_I$.

(7) If $x \in X$ or $x \in Y$, then $x \in X \cup Y$.

(8) $x \in X \cap Y$ iff $x \in X$ and $x \in Y$.

(9) If $x \in X$ and $X \subseteq Y$, then $x \in Y$.

(10) If $x \in X$ and $x \in Y$, then X overlaps Y.

(11) If X overlaps Y, then there exists x such that $x \in X$ and $x \in Y$.

(12) If $x \in X \setminus Y$, then $x \in X$.

2. Lattice Properties of Many Sorted Sets

One can prove the following proposition

(13) $X \subseteq X$.

Let us consider I, X, Y. Let us observe that $X = Y$ if and only if:

(Def.13) $X \subseteq Y$ and $Y \subseteq X$.

Next we state a number of propositions:

(14) If $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$.

(15) If $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$.

(16) $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$.

(17) $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$.

(18) If $X \subseteq Z$ and $Y \subseteq Z$, then $X \cup Y \subseteq Z$.

(19) If $Z \subseteq X$ and $Z \subseteq Y$, then $Z \subseteq X \cap Y$.

(20) If $X \subseteq Y$, then $X \cup Z \subseteq Y \cup Z$ and $Z \cup X \subseteq Z \cup Y$.

(21) If $X \subseteq Y$, then $X \cap Z \subseteq Y \cap Z$ and $Z \cap X \subseteq Z \cap Y$.

(22) If $X \subseteq Y$ and $Z \subseteq V$, then $X \cup Z \subseteq Y \cup V$.

(23) If $X \subseteq Y$ and $Z \subseteq V$, then $X \cap Z \subseteq Y \cap V$.

(24) If $X \subseteq Y$, then $X \cup Y = Y$ and $Y \cup X = Y$.

(25) If $X \subseteq Y$, then $X \cap Y = X$ and $Y \cap X = X$.

(26) $X \cap Y \subseteq X \cup Z$.

(27) If $X \subseteq Z$, then $X \cup Y \cap Z = (X \cup Y) \cap Z$.

(28) $X = Y \cup Z$ iff $Y \subseteq X$ and $Z \subseteq X$ and for every V such that $Y \subseteq V$ and $Z \subseteq V$ holds $X \subseteq V$.

(29) $X = Y \cap Z$ iff $X \subseteq Y$ and $X \subseteq Z$ and for every V such that $V \subseteq Y$ and $V \subseteq Z$ holds $V \subseteq X$.

(30) $X \cup X = X$.

(31) $X \cap X = X$.

(32) $X \cup Y = Y \cup X$.

(33) $X \cap Y = Y \cap X$.

(34) $(X \cup Y) \cup Z = X \cup (Y \cup Z)$.

(35) $(X \cap Y) \cap Z = X \cap (Y \cap Z)$.

(36) $X \cap (X \cup Y) = X$ and $(X \cup Y) \cap X = X$ and $X \cap (Y \cup X) = X$ and $(Y \cup X) \cap X = X$.

(37) $X \cup X \cap Y = X$ and $X \cap Y \cup X = X$ and $X \cup Y \cap X = X$ and $Y \cap X \cup X = X$.

(38) $X \cap (Y \cup Z) = X \cap Y \cup X \cap Z$ and $(Y \cup Z) \cap X = Y \cap X \cup Z \cap X$.

(39) $X \cup Y \cap Z = (X \cup Y) \cap (X \cup Z)$ and $Y \cap Z \cup X = (Y \cup X) \cap (Z \cup X)$.

(40) If $X \cap Y \cup X \cap Z = X$, then $X \subseteq Y \cup Z$.

(41) If $(X \cup Y) \cap (X \cup Z) = X$, then $Y \cap Z \subseteq X$.

(42) $X \cap Y \cup Y \cap Z \cup Z \cap X = (X \cup Y) \cap (Y \cup Z) \cap (Z \cup X)$.

(43) If $X \cup Y \subseteq Z$, then $X \subseteq Z$ and $Y \subseteq Z$.

(44) If $X \subseteq Y \cap Z$, then $X \subseteq Y$ and $X \subseteq Z$.

(45) $(X \cup Y) \cup Z = X \cup Z \cup (Y \cup Z)$ and $X \cup (Y \cup Z) = (X \cup Y) \cup (X \cup Z)$.

(46) $(X \cap Y) \cap Z = X \cap Z \cap (Y \cap Z)$ and $X \cap (Y \cap Z) = (X \cap Y) \cap (X \cap Z)$.

(47) $X \cup (X \cup Y) = X \cup Y$ and $X \cup Y \cup Y = X \cup Y$.

(48) $X \cap (X \cap Y) = X \cap Y$ and $X \cap Y \cap Y = X \cap Y$.

3. The Empty Many Sorted Set

Next we state several propositions:

(49) $\emptyset_I \subseteq X$.

(50) If $X \subseteq \emptyset_I$, then $X = \emptyset_I$.

(51) If $X \subseteq Y$ and $X \subseteq Z$ and $Y \cap Z = \emptyset_I$, then $X = \emptyset_I$.

(52) If $X \subseteq Y$ and $Y \cap Z = \emptyset_I$, then $X \cap Z = \emptyset_I$.

(53) $X \cup \emptyset_I = X$ and $\emptyset_I \cup X = X$.

(54) If $X \cup Y = \emptyset_I$, then $X = \emptyset_I$ and $Y = \emptyset_I$.

(55) $X \cap \emptyset_I = \emptyset_I$ and $\emptyset_I \cap X = \emptyset_I$.

(56) If $X \subseteq Y \cup Z$ and $X \cap Z = \emptyset_I$, then $X \subseteq Y$.

(57) If $Y \subseteq X$ and $X \cap Y = \emptyset_I$, then $Y = \emptyset_I$.

4. The Difference and the Symmetric Difference

We now state a number of propositions:

(58) $X \setminus Y = \emptyset_I$ iff $X \subseteq Y$.

(59) If $X \subseteq Y$, then $X \setminus Z \subseteq Y \setminus Z$.

(60) If $X \subseteq Y$, then $Z \setminus Y \subseteq Z \setminus X$.

(61) If $X \subseteq Y$ and $Z \subseteq V$, then $X \setminus V \subseteq Y \setminus Z$.

(62) $X \setminus Y \subseteq X$.

(63) If $X \subseteq Y \setminus X$, then $X = \emptyset_I$.

(64) $X \setminus X = \emptyset_I$.

(65) $X \setminus \emptyset_I = X$.

(66) $\emptyset_I \setminus X = \emptyset_I$.

(67) $X \setminus (X \cup Y) = \emptyset_I$ and $X \setminus (Y \cup X) = \emptyset_I$.

(68) $X \cap (Y \setminus Z) = X \cap Y \setminus Z$.

(69) $(X \setminus Y) \cap Y = \emptyset_I$ and $Y \cap (X \setminus Y) = \emptyset_I$.

(70) $X \setminus (Y \setminus Z) = (X \setminus Y) \cup X \cap Z$.

(71) $(X \setminus Y) \cup X \cap Y = X$ and $X \cap Y \cup (X \setminus Y) = X$.

(72) If $X \subseteq Y$, then $Y = X \cup (Y \setminus X)$ and $Y = (Y \setminus X) \cup X$.

(73) $X \cup (Y \setminus X) = X \cup Y$ and $(Y \setminus X) \cup X = Y \cup X$.

(74) $X \setminus (X \setminus Y) = X \cap Y$.

(75) $X \setminus Y \cap Z = (X \setminus Y) \cup (X \setminus Z)$.

(76) $X \setminus X \cap Y = X \setminus Y$ and $X \setminus Y \cap X = X \setminus Y$.

(77) $X \cap Y = \emptyset_I$ iff $X \setminus Y = X$.

(78) $(X \cup Y) \setminus Z = (X \setminus Z) \cup (Y \setminus Z)$.

(79) $X \setminus Y \setminus Z = X \setminus (Y \cup Z)$.

(80) $X \cap Y \setminus Z = (X \setminus Z) \cap (Y \setminus Z)$.

(81) $(X \cup Y) \setminus Y = X \setminus Y$.

(82) If $X \subseteq Y \cup Z$, then $X \setminus Y \subseteq Z$ and $X \setminus Z \subseteq Y$.

(83) $(X \cup Y) \setminus X \cap Y = (X \setminus Y) \cup (Y \setminus X)$.

(84) $X \setminus Y \setminus Y = X \setminus Y$.

(85) $X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z)$.

(86) If $X \setminus Y = Y \setminus X$, then $X = Y$.

(87) $X \cap (Y \setminus Z) = X \cap Y \setminus X \cap Z$ and $(Y \setminus Z) \cap X = Y \cap X \setminus Z \cap X$.

(88) If $X \setminus Y \subseteq Z$, then $X \subseteq Y \cup Z$.

(89) $X \setminus Y \subseteq X \dot{-} Y$.

(90) $X \dot{-} Y = (X \setminus Y) \cup (Y \setminus X)$.

(91) $X \dot{-} \emptyset_I = X$ and $\emptyset_I \dot{-} X = X$.

(92) $X \dot{-} X = \emptyset_I$.

(93) $X \dot{-} Y = Y \dot{-} X$.
(94) $X \cup Y = (X \dot{-} Y) \cup X \cap Y$.
(95) $X \dot{-} Y = (X \cup Y) \setminus X \cap Y$.
(96) $(X \dot{-} Y) \setminus Z = (X \setminus (Y \cup Z)) \cup (Y \setminus (X \cup Z))$.
(97) $X \setminus (Y \dot{-} Z) = (X \setminus (Y \cup Z)) \cup X \cap Y \cap Z$.
(98) $(X \dot{-} Y) \dot{-} Z = X \dot{-} (Y \dot{-} Z)$.
(99) If $X \setminus Y \subseteq Z$ and $Y \setminus X \subseteq Z$, then $X \dot{-} Y \subseteq Z$.
(100) $X \cup Y = X \dot{-} (Y \setminus X)$.
(101) $X \cap Y = X \dot{-} (X \setminus Y)$.
(102) $X \setminus Y = X \dot{-} X \cap Y$.
(103) $Y \setminus X = X \dot{-} (X \cup Y)$.
(104) $X \cup Y = X \dot{-} Y \dot{-} X \cap Y$.
(105) $X \cap Y = X \dot{-} Y \dot{-} (X \cup Y)$.

5. Meeting and Overlapping

The following propositions are true:

(106) If X overlaps Y or X overlaps Z, then X overlaps $Y \cup Z$.
(107) If X overlaps Y, then Y overlaps X.
(108) If X overlaps Y and $Y \subseteq Z$, then X overlaps Z.
(109) If X overlaps Y and $X \subseteq Z$, then Z overlaps Y.
(110) If $X \subseteq Y$ and $Z \subseteq V$ and X overlaps Z, then Y overlaps V.
(111) If X overlaps $Y \cap Z$, then X overlaps Y and X overlaps Z.
(112) If X overlaps Z and $X \subseteq V$, then X overlaps $Z \cap V$.
(113) If X overlaps $Y \setminus Z$, then X overlaps Y.
(114) If Y does not overlap Z, then $X \cap Y$ does not overlap $X \cap Z$ and $Y \cap X$ does not overlap $Z \cap X$.
(115) If X overlaps $Y \setminus Z$, then Y overlaps $X \setminus Z$.
(116) If X meets Y and $Y \subseteq Z$, then X meets Z.
(117) If X meets Y, then Y meets X.
(118) Y misses $X \setminus Y$.
(119) $X \cap Y$ misses $X \setminus Y$.
(120) $X \cap Y$ misses $X \dot{-} Y$.
(121) If X misses Y, then $X \cap Y = \emptyset_I$.
(122) If $X \neq \emptyset_I$, then X meets X.
(123) If $X \subseteq Y$ and $X \subseteq Z$ and Y misses Z, then $X = \emptyset_I$.
(124) If $Z \cup V = X \cup Y$ and X misses Z and Y misses V, then $X = V$ and $Y = Z$.

(125) If $Z \cup V = X \cup Y$ and Y misses Z and X misses V, then $X = Z$ and $Y = V$.

(126) If X misses Y, then $X \setminus Y = X$ and $Y \setminus X = Y$.

(127) If X misses Y, then $(X \cup Y) \setminus Y = X$ and $(X \cup Y) \setminus X = Y$.

(128) If $X \setminus Y = X$, then X misses Y and Y misses X.

(129) $X \setminus Y$ misses $Y \setminus X$.

6. The Second Inclusion

Let us consider I, X, Y. The predicate $X \sqsubseteq Y$ is defined as follows:

(Def.14) For every x such that $x \in X$ holds $x \in Y$.

The following three propositions are true:

(130) If $X \subseteq Y$, then $X \sqsubseteq Y$.

(131) $X \sqsubseteq X$.

(132) If $X \sqsubseteq Y$ and $Y \sqsubseteq Z$, then $X \sqsubseteq Z$.

7. Non Empty and Non-empty Many Sorted Sets

The following propositions are true:

(133) $\emptyset_\emptyset \in \emptyset_\emptyset$.

(134) For every many sorted set X of $\emptyset$ holds $X = \emptyset$.

We follow a convention: I will be a non empty set and x, X, Y, Z will be many sorted sets of I.

The following propositions are true:

(135) If X overlaps Y, then X meets Y.

(136) It is not true that there exists x such that $x \in \emptyset_I$.

(137) If $x \in X$ and $x \in Y$, then $X \cap Y \neq \emptyset_I$.

(138) X does not overlap $\emptyset_I$ and $\emptyset_I$ does not overlap X.

(139) If $X \cap Y = \emptyset_I$, then X does not overlap Y.

(140) If X overlaps X, then $X \neq \emptyset_I$.

Let I be a set. A many sorted set of I is empty yielding if:

(Def.15) For every i such that $i \in I$ holds it(i) is empty.

A many sorted set of I is non empty set yielding if:

(Def.16) For every i such that $i \in I$ holds it(i) is non empty.

Let I be a non empty set. Observe that every many sorted set of I which is non-empty is also non empty and every many sorted set of I which is empty is also non non-empty.

One can prove the following propositions:

(141) X is empty iff $X = \emptyset_I$.

(142) If Y is empty and $X \subseteq Y$, then X is empty.

(143) If X is non-empty and $X \subseteq Y$, then Y is non-empty.

(144) If X is non-empty and $X \sqsubseteq Y$, then $X \subseteq Y$.

(145) If X is non-empty and $X \sqsubseteq Y$, then Y is non-empty.

In the sequel X denotes a non-empty many sorted set of I.

The following propositions are true:

(146) There exists x such that $x \in X$.

(147) If for every x holds $x \in X$ iff $x \in Y$, then $X = Y$.

(148) If for every x holds $x \in X$ iff $x \in Y$ and $x \in Z$, then $X = Y \cap Z$.

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Subalgebras of the Universal Algebra. Lattices of Subalgebras

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Summary. Introduces a definition of a subalgebra of a universal algebra. A notion of similar algebras and basic operations on subalgebras such as a subalgebra generated by a set, the intersection and the sum of two subalgebras were introduced. Some basic facts concerning the above notions have been proved. The article also contains the definition of a lattice of subalgebras of a universal algebra.

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The papers [7], [8], [4], [1], [5], [3], [9], [2], and [6] provide the terminology and notation for this paper.

One can prove the following propositions:

(1) For every natural number n and for every non empty set D and for every non empty subset D_1 of D holds $D^n \cap {D_1}^n = {D_1}^n$.

(2) For every non empty set D and for every homogeneous quasi total non empty partial function h from D^* to D holds $\operatorname{dom} h = D^{\operatorname{arity} h}$.

We follow a convention: U_0, U_1, U_2, U_3 denote universal algebras, n, i denote natural numbers, and a denotes an element of the carrier of U_0.

Let D be a non empty set. A non empty set is called a set of universal functions on D if:

(Def.1) Every element of it is a homogeneous quasi total non empty partial function from D^* to D.

Let D be a non empty set and let P be a set of universal functions on D. We see that the element of P is a homogeneous quasi total non empty partial function from D^* to D.

Let us consider U_1. A set of universal functions on U_1 is a set of universal functions on the carrier of U_1.

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Let U_1 be a universal algebra structure. A partial function on U_1 is a partial function from (the carrier of U_1)* to the carrier of U_1.

Let us consider U_1, U_2. We say that U_1 and U_2 are similar if and only if:

(Def.2) signature U_1 = signature U_2.

Let us observe that the predicate introduced above is reflexive symmetric.

The following propositions are true:

(3) If U_1 and U_2 are similar, then len Opers U_1 = len Opers U_2.

(4) If U_1 and U_2 are similar and U_2 and U_3 are similar, then U_1 and U_3 are similar.

(5) rng Opers U_0 is a non empty subset of (the carrier of U_0)$^* \dot{\rightarrow}$the carrier of U_0.

Let us consider U_0. The functor Operations(U_0) yielding a set of universal functions on U_0 is defined as follows:

(Def.3) Operations(U_0) = rng Opers U_0.

Let us consider U_1. A operation of U_1 is an element of Operations(U_1).

Let us consider U_0. A subset of U_0 is a subset of the carrier of U_0.

In the sequel x_1, y_1 will denote finite sequences of elements of A.

One can prove the following proposition

(6) If $n \in$ dom Opers U_0, then (Opers U_0)(n) is a operation of U_0.

Let U_0 be a universal algebra, let A be a subset of U_0, and let o be a operation of U_0. We say that A is closed on o if and only if:

(Def.4) For every finite sequence s of elements of A such that len s = arity o holds $o(s) \in A$.

Let U_0 be a universal algebra and let A be a subset of U_0. We say that A is operations closed if and only if:

(Def.5) For every operation o of U_0 holds A is closed on o.

Let us consider U_0, A, o. Let us assume that A is closed on o. The functor o_A yielding a homogeneous quasi total non empty partial function from A^* to A is defined as follows:

(Def.6) $o_A = o \restriction A^{\mathrm{arity}\, o}$.

Let us consider U_0, A. The functor Opers(U_0, A) yields a finite sequence of elements of $A^* \dot{\rightarrow} A$ and is defined as follows:

(Def.7) dom Opers(U_0, A) = dom Opers U_0 and for all n, o such that $n \in$ dom Opers(U_0, A) and o = (Opers U_0)(n) holds (Opers(U_0, A))(n) = o_A.

The following two propositions are true:

(7) For every non empty subset B of U_0 such that B = the carrier of U_0 holds B is operations closed and for every o holds $o_B = o$.

(8) Let U_1 be a universal algebra, and let A be a non empty subset of U_1, and let o be a operation of U_1. If A is closed on o, then arity(o_A) = arity o.

Let us consider U_0. A universal algebra is said to be a subalgebra of U_0 if it satisfies the conditions (Def.8).

(Def.8) (i) The carrier of it is a subset of U_0, and

(ii) for every non empty subset B of U_0 such that $B =$ the carrier of it holds Opers it $=$ Opers(U_0, B) and B is operations closed.

Let U_0 be a universal algebra. One can verify that there exists a subalgebra of U_0 which is strict.

One can prove the following propositions:

(9) Let U_0, U_1 be universal algebras, and let o_0 be a operation of U_0, and let o_1 be a operation of U_1, and let n be a natural number. Suppose U_0 is a subalgebra of U_1 and $n \in$ dom Opers U_0 and $o_0 =$ (Opers U_0)(n) and $o_1 =$ (Opers U_1)(n). Then arity $o_0 =$ arity o_1.

(10) If U_0 is a subalgebra of U_1, then dom Opers $U_0 =$ dom Opers U_1.

(11) U_0 is a subalgebra of U_0.

(12) If U_0 is a subalgebra of U_1 and U_1 is a subalgebra of U_2, then U_0 is a subalgebra of U_2.

(13) If U_1 is a strict subalgebra of U_2 and U_2 is a strict subalgebra of U_1, then $U_1 = U_2$.

(14) For all subalgebras U_1, U_2 of U_0 such that the carrier of $U_1 \subseteq$ the carrier of U_2 holds U_1 is a subalgebra of U_2.

(15) For all strict subalgebra U_1, U_2 of U_0 such that the carrier of $U_1 =$ the carrier of U_2 holds $U_1 = U_2$.

(16) If U_1 is a subalgebra of U_2, then U_1 and U_2 are similar.

(17) For every non empty subset A of U_0 holds $\langle A, \mathrm{Opers}(U_0, A)\rangle$ is a strict universal algebra.

Let U_0 be a universal algebra and let A be a non empty subset of U_0. Let us assume that A is operations closed. The functor $\langle A, \mathrm{Ops}\rangle$ yielding a strict subalgebra of U_0 is defined as follows:

(Def.9) $\langle A, \mathrm{Ops}\rangle = \langle A, \mathrm{Opers}(U_0, A)\rangle$.

Let us consider U_0 and let U_1, U_2 be subalgebras of U_0. Let us assume that (the carrier of U_1) $\cap$ (the carrier of U_2) $\neq \emptyset$. The functor $U_1 \cap U_2$ yielding a strict subalgebra of U_0 is defined by the conditions (Def.10).

(Def.10) (i) The carrier of $U_1 \cap U_2 =$ (the carrier of U_1) $\cap$ (the carrier of U_2), and

(ii) for every non empty subset B of U_0 such that $B =$ the carrier of $U_1 \cap U_2$ holds Opers($U_1 \cap U_2$) $=$ Opers(U_0, B) and B is operations closed.

Let us consider U_0. The functor Constants(U_0) yielding a subset of U_0 is defined by:

(Def.11) Constants(U_0) $= \{a : a$ ranges over elements of the carrier of U_0, $\exists_o$ arity $o = 0 \land a \in \mathrm{rng}\, o\}$.

A universal algebra has constants if:

(Def.12) There exists a operation o of it such that arity $o = 0$.

Let us note that there exists a universal algebra which is strict and has constants.

Let U_0 be a universal algebra with constants. Then Constants(U_0) is a non empty subset of U_0.

One can prove the following three propositions:

(18) For every universal algebra U_0 and for every subalgebra U_1 of U_0 holds Constants(U_0) is a subset of U_1.

(19) For every universal algebra U_0 with constants and for every subalgebra U_1 of U_0 holds Constants(U_0) is a non empty subset of U_1.

(20) Let U_0 be a universal algebra with constants and let U_1, U_2 be subalgebras of U_0. Then (the carrier of U_1) $\cap$ (the carrier of U_2) $\neq \emptyset$.

Let U_0 be a universal algebra and let A be a subset of U_0. Let us assume that Constants(U_0) $\neq \emptyset$ or $A \neq \emptyset$. The functor $\mathrm{Gen}^{\mathrm{UA}}(A)$ yields a strict subalgebra of U_0 and is defined by the conditions (Def.13).

(Def.13) (i) $A \subseteq$ the carrier of $\mathrm{Gen}^{\mathrm{UA}}(A)$, and

(ii) for every subalgebra U_1 of U_0 such that $A \subseteq$ the carrier of U_1 holds $\mathrm{Gen}^{\mathrm{UA}}(A)$ is a subalgebra of U_1.

Next we state two propositions:

(21) For every strict universal algebra U_0 holds $\mathrm{Gen}^{\mathrm{UA}}(\Omega_{\text{the carrier of } U_0}) = U_0$.

(22) Let U_0 be a universal algebra, and let U_1 be a strict subalgebra of U_0, and let B be a non empty subset of U_0. If $B =$ the carrier of U_1, then $\mathrm{Gen}^{\mathrm{UA}}(B) = U_1$.

Let U_0 be a universal algebra and let U_1, U_2 be subalgebras of U_0. The functor $U_1 \bigsqcup U_2$ yields a strict subalgebra of U_0 and is defined by:

(Def.14) For every non empty subset A of U_0 such that $A =$ (the carrier of U_1) $\cup$ (the carrier of U_2) holds $U_1 \bigsqcup U_2 = \mathrm{Gen}^{\mathrm{UA}}(A)$.

Next we state four propositions:

(23) Let U_0 be a universal algebra, and let U_1 be a subalgebra of U_0, and let A, B be subsets of U_0. If $A \neq \emptyset$ or Constants(U_0) $\neq \emptyset$ and if $B = A \cup$ the carrier of U_1, then $\mathrm{Gen}^{\mathrm{UA}}(A) \bigsqcup U_1 = \mathrm{Gen}^{\mathrm{UA}}(B)$.

(24) For every universal algebra U_0 and for all subalgebras U_1, U_2 of U_0 holds $U_1 \bigsqcup U_2 = U_2 \bigsqcup U_1$.

(25) For every universal algebra U_0 with constants and for all strict subalgebra U_1, U_2 of U_0 holds $U_1 \cap (U_1 \bigsqcup U_2) = U_1$.

(26) For every universal algebra U_0 with constants and for all strict subalgebra U_1, U_2 of U_0 holds $U_1 \cap U_2 \bigsqcup U_2 = U_2$.

Let U_0 be a universal algebra. The functor Subalgebras(U_0) yields a non empty set and is defined as follows:

(Def.15) For every x holds $x \in$ Subalgebras(U_0) iff x is a strict subalgebra of U_0.

Let U_0 be a universal algebra. The functor $\bigsqcup_{U_0}$ yielding a binary operation on Subalgebras(U_0) is defined by:

(Def.16) For all elements x, y of Subalgebras(U_0) and for all strict subalgebra U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds $\bigsqcup_{(U_0)}(x, y) = U_1 \bigsqcup U_2$.

Let U_0 be a universal algebra. The functor $\sqcap_{U_0}$ yields a binary operation on Subalgebras(U_0) and is defined by:

(Def.17) For all elements x, y of Subalgebras(U_0) and for all strict subalgebra U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds $\sqcap_{(U_0)}(x, y) = U_1 \cap U_2$.

One can prove the following four propositions:

(27) $\sqcup_{(U_0)}$ is commutative.

(28) $\sqcup_{(U_0)}$ is associative.

(29) For every universal algebra U_0 with constants holds $\sqcap_{(U_0)}$ is commutative.

(30) For every universal algebra U_0 with constants holds $\sqcap_{(U_0)}$ is associative.

Let U_0 be a universal algebra with constants. The lattice of subalgebras of U_0 yielding a strict lattice is defined as follows:

(Def.18) The lattice of subalgebras of $U_0 = \langle \text{Subalgebras}(U_0), \sqcup_{(U_0)}, \sqcap_{(U_0)} \rangle$.

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Hahn-Banach Theorem

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Summary. We prove a version of Hahn-Banach Theorem.

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The notation and terminology used here are introduced in the following papers: [13], [5], [9], [2], [3], [17], [16], [15], [8], [4], [10], [6], [14], [12], [11], [1], and [7].

1. Preliminaries

The following propositions are true:

(1) For arbitrary x, y and for every function f such that $\langle x, y\rangle \in f$ holds $y \in \operatorname{rng} f$.

(2) For every set X and for all functions f, g such that $X \subseteq \operatorname{dom} f$ and $f \subseteq g$ holds $f \restriction X = g \restriction X$.

(3) For every non empty set A and for arbitrary b such that $A \neq \{b\}$ there exists an element a of A such that $a \neq b$.

Let B be a non empty functional set. Observe that every element of B is function-like.

The following propositions are true:

(4) For all sets X, Y holds every non empty subset of $X \dot{\to} Y$ is a non empty functional set.

(5) Let B be a non empty functional set and let f be a function. Suppose $f = \bigcup B$. Then $\operatorname{dom} f = \bigcup\{\operatorname{dom} g : g \text{ ranges over elements of } B, \}$ and $\operatorname{rng} f = \bigcup\{\operatorname{rng} g : g \text{ ranges over elements of } B, \}$.

The scheme *NonUniqExD'* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

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There exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element e of $\mathcal{A}$ holds $\mathcal{P}[e, f(e)]$

provided the parameters satisfy the following condition:

- For every element e of $\mathcal{A}$ there exists an element u of $\mathcal{B}$ such that $\mathcal{P}[e, u]$.

One can prove the following propositions:

(6) For every non empty subset A of $\overline{\mathbb{R}}$ such that for every *Real number* r such that $r \in A$ holds $r \leq -\infty$ holds $A = \{-\infty\}$.

(7) For every non empty subset A of $\overline{\mathbb{R}}$ such that for every *Real number* r such that $r \in A$ holds $+\infty \leq r$ holds $A = \{+\infty\}$.

(8) Let A be a non empty subset of $\overline{\mathbb{R}}$ and let r be a *Real number.* If $r < \sup A$, then there exists a *Real number* s such that $s \in A$ and $r < s$.

(9) Let A be a non empty subset of $\overline{\mathbb{R}}$ and let r be a *Real number.* If $\inf A < r$, then there exists a *Real number* s such that $s \in A$ and $s < r$.

(10) Let A, B be non empty subset of $\overline{\mathbb{R}}$. Suppose that for all *Real numbers* r, s such that $r \in A$ and $s \in B$ holds $r \leq s$. Then $\sup A \leq \inf B$.

(12)[1] Let x, y be real numbers and let x', y' be *Real numbers.* If $x = x'$ and $y = y'$, then $x \leq y$ iff $x' \leq y'$.

2. Sets Linearly Ordered by the Inclusion

A set is $\subseteq$ -linear if:

(Def.1) For arbitrary x, y such that $x \in$ it and $y \in$ it holds $x \subseteq y$ or $y \subseteq x$.

Let A be a non empty set. Note that there exists a subset of A which is $\subseteq$ -linear and non empty.

We now state the proposition

(13) For all sets X, Y and for every $\subseteq$ -linear non empty subset B of $X \dot{\rightarrow} Y$ holds $\bigcup B \in X \dot{\rightarrow} Y$.

3. Subspaces of a Real Linear Space

In the sequel V will be a real linear space.

One can prove the following propositions:

(14) For all subspaces W_1, W_2 of V holds the carrier of $W_1 \subseteq$ the carrier of $W_1 + W_2$.

(15) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$, then $v \lhd (W_1, W_2) = \langle v_1, v_2 \rangle$.

[1]The proposition (11) has been removed.

(16) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v \lhd (W_1, W_2) = \langle v_1, v_2 \rangle$, then $v = v_1 + v_2$.

(17) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v \lhd (W_1, W_2) = \langle v_1, v_2 \rangle$, then $v_1 \in W_1$ and $v_2 \in W_2$.

(18) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v \lhd (W_1, W_2) = \langle v_1, v_2 \rangle$, then $v \lhd (W_2, W_1) = \langle v_2, v_1 \rangle$.

(19) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v be a vector of V. If $v \in W_1$, then $v \lhd (W_1, W_2) = \langle v, 0_V \rangle$.

(20) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v be a vector of V. If $v \in W_2$, then $v \lhd (W_1, W_2) = \langle 0_V, v \rangle$.

(21) Let V_1 be a subspace of V, and let W_1 be a subspace of V_1, and let v be a vector of V. If $v \in W_1$, then v is a vector of V_1.

(22) For all subspaces V_1, V_2, W of V and for all subspaces W_1, W_2 of W such that $W_1 = V_1$ and $W_2 = V_2$ holds $W_1 + W_2 = V_1 + V_2$.

(23) For every subspace W of V and for every vector v of V and for every vector w of W such that $v = w$ holds $\mathrm{Lin}(\{w\}) = \mathrm{Lin}(\{v\})$.

(24) Let v be a vector of V and let X be a subspace of V. Suppose $v \notin X$. Let y be a vector of $X + \mathrm{Lin}(\{v\})$ and let W be a subspace of $X + \mathrm{Lin}(\{v\})$. If $v = y$ and $W = X$, then $X + \mathrm{Lin}(\{v\})$ is the direct sum of W and $\mathrm{Lin}(\{y\})$.

(25) Let v be a vector of V, and let X be a subspace of V, and let y be a vector of $X + \mathrm{Lin}(\{v\})$, and let W be a subspace of $X + \mathrm{Lin}(\{v\})$. If $v = y$ and $X = W$ and $v \notin X$, then $y \lhd (W, \mathrm{Lin}(\{y\})) = \langle 0_W, y \rangle$.

(26) Let v be a vector of V, and let X be a subspace of V, and let y be a vector of $X + \mathrm{Lin}(\{v\})$, and let W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$. If $w \in X$, then $w \lhd (W, \mathrm{Lin}(\{y\})) = \langle w, 0_V \rangle$.

(27) For every vector v of V and for all subspaces W_1, W_2 of V there exist vectors v_1, v_2 of V such that $v \lhd (W_1, W_2) = \langle v_1, v_2 \rangle$.

(28) Let v be a vector of V, and let X be a subspace of V, and let y be a vector of $X + \mathrm{Lin}(\{v\})$, and let W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$. Then there exists a vector x of X and there exists a real number r such that $w \lhd (W, \mathrm{Lin}(\{y\})) = \langle x, r \cdot v \rangle$.

(29) Let v be a vector of V, and let X be a subspace of V, and let y be a vector of $X + \mathrm{Lin}(\{v\})$, and let W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w_1, w_2 be vectors of $X + \mathrm{Lin}(\{v\})$, and let x_1, x_2 be vectors of X, and let r_1, r_2 be real numbers. If $w_1 \lhd (W, \mathrm{Lin}(\{y\})) = \langle x_1, r_1 \cdot v \rangle$ and $w_2 \lhd (W, \mathrm{Lin}(\{y\})) = \langle x_2, r_2 \cdot v \rangle$, then $(w_1 + w_2) \lhd (W, \mathrm{Lin}(\{y\})) = \langle x_1 + x_2, (r_1 + r_2) \cdot v \rangle$.

(30) Let v be a vector of V, and let X be a subspace of V, and let y be a vector of $X + \mathrm{Lin}(\{v\})$, and let W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$, and let x be a vector of X, and let t, r be real numbers. If $w \lhd (W, \mathrm{Lin}(\{y\})) = \langle x, r \cdot v \rangle$, then $(t \cdot w) \lhd (W, \mathrm{Lin}(\{y\})) = \langle t \cdot x, t \cdot r \cdot v \rangle$.

4. Functionals

Let V be an RLS structure.

(Def.2) A function from the carrier of V into $\mathbb{R}$ is called a functional in V.

Let us consider V. A functional in V is subadditive if:

(Def.3) For all vectors x, y of V holds $\mathrm{it}(x + y) \leq \mathrm{it}(x) + \mathrm{it}(y)$.

A functional in V is additive if:

(Def.4) For all vectors x, y of V holds $\mathrm{it}(x + y) = \mathrm{it}(x) + \mathrm{it}(y)$.

A functional in V is homogeneous if:

(Def.5) For every vector x of V and for every real number r holds $\mathrm{it}(r \cdot x) = r \cdot \mathrm{it}(x)$.

A functional in V is positively homogeneous if:

(Def.6) For every vector x of V and for every real number r such that $r > 0$ holds $\mathrm{it}(r \cdot x) = r \cdot \mathrm{it}(x)$.

A functional in V is semi-homogeneous if:

(Def.7) For every vector x of V and for every real number r such that $r \geq 0$ holds $\mathrm{it}(r \cdot x) = r \cdot \mathrm{it}(x)$.

A functional in V is absolutely homogeneous if:

(Def.8) For every vector x of V and for every real number r holds $\mathrm{it}(r \cdot x) = |r| \cdot \mathrm{it}(x)$.

A functional in V is 0-preserving if:

(Def.9) $\mathrm{It}(0_V) = 0$.

Let us consider V. One can verify the following observations:

* every functional in V which is additive is also subadditive,
* every functional in V which is homogeneous is also positively homogeneous,
* every functional in V which is semi-homogeneous is also positively homogeneous,
* every functional in V which is semi-homogeneous is also 0-preserving,
* every functional in V which is absolutely homogeneous is also semi-homogeneous, and
* every functional in V which is 0-preserving and positively homogeneous is also semi-homogeneous.

Let us consider V. Observe that there exists a functional in V which is additive absolutely homogeneous and homogeneous.

Let us consider V. A Banach functional in V is a subadditive positively homogeneous functional in V. A linear functional in V is an additive homogeneous functional in V.

We now state four propositions:

(31) For every homogeneous functional L in V and for every vector v of V holds $L(-v) = -L(v)$.

(32) For every linear functional L in V and for all vectors v_1, v_2 of V holds $L(v_1 - v_2) = L(v_1) - L(v_2)$.

(33) For every additive functional L in V holds $L(0_V) = 0$.

(34) Let X be a subspace of V, and let f_1 be a linear functional in X, and let v be a vector of V, and let y be a vector of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $v \notin X$. Let r be a real number. Then there exists a linear functional p_1 in $X + \mathrm{Lin}(\{v\})$ such that $p_1 \restriction (\text{the carrier of } X) = f_1$ and $p_1(y) = r$.

5. Hahn-Banach Theorem

One can prove the following three propositions:

(35) Let V be a real linear space, and let X be a subspace of V, and let q be a Banach functional in V, and let f_1 be a linear functional in X. Suppose that for every vector x of X and for every vector v of V such that $x = v$ holds $f_1(x) \leq q(v)$. Then there exists a linear functional p_1 in V such that $p_1 \restriction (\text{the carrier of } X) = f_1$ and for every vector x of V holds $p_1(x) \leq q(x)$.

(36) For every real normed space V holds the norm of V is an absolutely homogeneous subadditive functional in V.

(37) Let V be a real normed space, and let X be a subspace of V, and let f_1 be a linear functional in X. Suppose that for every vector x of X and for every vector v of V such that $x = v$ holds $f_1(x) \leq \|v\|$. Then there exists a linear functional p_1 in V such that $p_1 \restriction (\text{the carrier of } X) = f_1$ and for every vector x of V holds $p_1(x) \leq \|x\|$.

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Homomorphisms of Lattices, Finite Join and Finite Meet

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The articles [9], [4], [2], [3], [8], [10], [6], [1], [5], and [7] provide the terminology and notation for this paper.

1. Preliminaries

We adopt the following convention: X, X_1, X_2, Y, Z will denote sets and x will be arbitrary.

Next we state three propositions:

(1) If $\bigcup Y \subseteq Z$ and $X \in Y$, then $X \subseteq Z$.

(2) $\bigcup(X \Cap Y) = \bigcup X \cap \bigcup Y$.

(3) Given X. Suppose that

(i) $X \neq \emptyset$, and

(ii) for every Z such that $Z \neq \emptyset$ and $Z \subseteq X$ and for all X_1, X_2 such that $X_1 \in Z$ and $X_2 \in Z$ holds $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$ there exists Y such that $Y \in X$ and for every X_1 such that $X_1 \in Z$ holds $X_1 \subseteq Y$.

Then there exists Y such that $Y \in X$ and for every Z such that $Z \in X$ and $Z \neq Y$ holds $Y \not\subseteq Z$.

2. Lattice Theory

We adopt the following convention: L denotes a lattice, F, H denote filters of L, and p, q, r denote elements of the carrier of L.

One can prove the following propositions:

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(4) $[L)$ is prime.

(5) $F \subseteq [F \cup H)$ and $H \subseteq [F \cup H)$.

(6) If $p \in [[q) \cup F)$, then there exists r such that $r \in F$ and $q \sqcap r \sqsubseteq p$.

We adopt the following rules: L_1, L_2 will be lattices, a_1, b_1 will be elements of the carrier of L_1, and a_2 will be an element of the carrier of L_2.

Let us consider L_1, L_2. A function from the carrier of L_1 into the carrier of L_2 is called a homomorphism from L_1 to L_2 if:

(Def.1) $\mathrm{It}(a_1 \sqcup b_1) = \mathrm{it}(a_1) \sqcup \mathrm{it}(b_1)$ and $\mathrm{it}(a_1 \sqcap b_1) = \mathrm{it}(a_1) \sqcap \mathrm{it}(b_1)$.

In the sequel f is a homomorphism from L_1 to L_2.

We now state the proposition

(7) If $a_1 \sqsubseteq b_1$, then $f(a_1) \sqsubseteq f(b_1)$.

Let us consider L_1, L_2, f. We say that f is monomorphism if and only if:

(Def.2) f is one-to-one.

We say that f is epimorphism if and only if:

(Def.3) $\mathrm{rng}\, f =$ the carrier of L_2.

Next we state two propositions:

(8) If f is monomorphism, then $a_1 \sqsubseteq b_1$ iff $f(a_1) \sqsubseteq f(b_1)$.

(9) If f is epimorphism, then for every a_2 there exists a_1 such that $a_2 = f(a_1)$.

Let us consider L_1, L_2, f. We say that f is isomorphism if and only if:

(Def.4) f is monomorphism and epimorphism.

Let us consider L_1, L_2. We say that L_1 and L_2 are isomorphic if and only if:

(Def.5) There exists f which is isomorphism.

Let us consider L_1, L_2, f. We say that f preserves implication if and only if:

(Def.6) $f(a_1 \Rightarrow b_1) = f(a_1) \Rightarrow f(b_1)$.

We say that f preserves top if and only if:

(Def.7) $f(\top_{(L_1)}) = \top_{(L_2)}$.

We say that f preserves bottom if and only if:

(Def.8) $f(\bot_{(L_1)}) = \bot_{(L_2)}$.

We say that f preserves complement if and only if:

(Def.9) $f({a_1}^{\mathrm{c}}) = f(a_1)^{\mathrm{c}}$.

Let us consider L. A non empty subset of the carrier of L is said to be a closed subset of L if:

(Def.10) If $p \in$ it and $q \in$ it, then $p \sqcap q \in$ it and $p \sqcup q \in$ it.

Next we state two propositions:

(10) The carrier of L is a closed subset of L.

(11) Every filter of L is a closed subset of L.

Let L be a lattice. The functor id_L yields a function from the carrier of L into the carrier of L and is defined as follows:

(Def.11) $\mathrm{id}_L = \mathrm{id}_{(\text{the carrier of } L)}$.

Next we state two propositions:

(12) For every element b of the carrier of L holds $\mathrm{id}_L(b) = b$.

(13) For every function f from the carrier of L into the carrier of L holds $f \cdot \mathrm{id}_L = f$ and $\mathrm{id}_L \cdot f = f$.

In the sequel B denotes a finite subset of the carrier of L.

Let us consider L, B. The functor $\bigsqcup^{\mathrm{f}}_B$ yields an element of the carrier of L and is defined by:

(Def.12) $\bigsqcup^{\mathrm{f}}_B = \bigsqcup^{\mathrm{f}}_B(\mathrm{id}_L)$.

The functor $\bigsqcap^{\mathrm{f}}_B$ yielding an element of the carrier of L is defined by:

(Def.13) $\bigsqcap^{\mathrm{f}}_B = \bigsqcap^{\mathrm{f}}_B(\mathrm{id}_L)$.

The following propositions are true:

(14) $\bigsqcap^{\mathrm{f}}_B =$ (the meet operation of L)-$\sum_B \mathrm{id}_L$.

(15) $\bigsqcup^{\mathrm{f}}_B =$ (the join operation of L)-$\sum_B \mathrm{id}_L$.

(16) $\bigsqcup^{\mathrm{f}}_{\{p\}} = p$.

(17) $\bigsqcap^{\mathrm{f}}_{\{p\}} = p$.

3. Distributive Lattices

In the sequel D_1 denotes a distributive lattice and f denotes a homomorphism from D_1 to L_2.

One can prove the following proposition

(18) If f is epimorphism, then L_2 is distributive.

4. Lower-bounded Lattices

We adopt the following rules: ℓ_1 is a lower-bounded lattice, B, B_1, B_2 are finite subsets of the carrier of ℓ_1, and b is an element of the carrier of ℓ_1.

Next we state the proposition

(19) Let f be a homomorphism from ℓ_1 to L_2. If f is epimorphism, then L_2 is lower-bounded and f preserves bottom.

In the sequel f will be a unary operation on the carrier of ℓ_1.

We now state several propositions:

(20) $\bigsqcup^{\mathrm{f}}_{B \cup \{b\}} f = \bigsqcup^{\mathrm{f}}_B f \sqcup f(b)$.

(21) $\bigsqcup^{\mathrm{f}}_{B \cup \{b\}} = \bigsqcup^{\mathrm{f}}_B \sqcup b$.

(22) $\bigsqcup^{\mathrm{f}}_{(B_1)} \sqcup \bigsqcup^{\mathrm{f}}_{(B_2)} = \bigsqcup^{\mathrm{f}}_{B_1 \cup B_2}$.

(23) $\bigsqcup^{\mathrm{f}}_{\emptyset_{\text{the carrier of } \ell_1}} = \perp_{(\ell_1)}$.

(24) For every closed subset A of ℓ_1 such that $\bot_{(\ell_1)} \in A$ and for every B such that $B \subseteq A$ holds $\bigsqcup^{\mathrm{f}}_{B} \in A$.

5. Upper-bounded Lattices

We adopt the following rules: ℓ_2 will denote an upper-bounded lattice, B, B_1, B_2 will denote finite subsets of the carrier of ℓ_2, and b will denote an element of the carrier of ℓ_2.

One can prove the following two propositions:

(25) For every homomorphism f from ℓ_2 to L_2 such that f is epimorphism holds L_2 is upper-bounded and f preserves top.

(26) $\bigsqcap^{\mathrm{f}}_{\emptyset_{\text{the carrier of } \ell_2}} = \top_{(\ell_2)}$.

In the sequel f, g will be unary operations on the carrier of ℓ_2.

The following propositions are true:

(27) $\bigsqcap^{\mathrm{f}}_{B \cup \{b\}} f = \bigsqcap^{\mathrm{f}}_{B} f \sqcap f(b)$.

(28) $\bigsqcap^{\mathrm{f}}_{B \cup \{b\}} = \bigsqcap^{\mathrm{f}}_{B} \sqcap b$.

(29) $\bigsqcap^{\mathrm{f}}_{f \circ B} g = \bigsqcap^{\mathrm{f}}_{B} (g \cdot f)$.

(30) $\bigsqcap^{\mathrm{f}}_{(B_1)} \sqcap \bigsqcap^{\mathrm{f}}_{(B_2)} = \bigsqcap^{\mathrm{f}}_{B_1 \cup B_2}$.

(31) For every closed subset F of ℓ_2 such that $\top_{(\ell_2)} \in F$ and for every B such that $B \subseteq F$ holds $\bigsqcap^{\mathrm{f}}_{B} \in F$.

6. Distributive Upper-bounded Lattices

In the sequel D_1 will be a distributive upper-bounded lattice, B will be a finite subset of the carrier of D_1, and p will be an element of the carrier of D_1.

Next we state the proposition

(32) $\bigsqcap^{\mathrm{f}}_{B} \sqcup p = \bigsqcap^{\mathrm{f}}_{((\text{the join operation of } D_1)^{\circ}(\mathrm{id}_{(D_1)}, p))^{\circ} B}$.

7. Implicative Lattices

For simplicity we adopt the following rules: C_1 denotes a complemented lattice, I_1 denotes an implicative lattice, f denotes a homomorphism from I_1 to C_1, and i, j, k denote elements of the carrier of I_1.

The following propositions are true:

(33) $f(i) \sqcap f(i \Rightarrow j) \sqsubseteq f(j)$.

(34) If f is monomorphism, then if $f(i) \sqcap f(k) \sqsubseteq f(j)$, then $f(k) \sqsubseteq f(i \Rightarrow j)$.

(35) If f is isomorphism, then C_1 is implicative and f preserves implication.

8. Boolean Lattices

For simplicity we adopt the following rules: B_3 will be a Boolean lattice, f will be a homomorphism from B_3 to C_1, A will be a non empty subset of the carrier of B_3, a, b, c, p, q will be elements of the carrier of B_3, and B, B_0 will be finite subsets of the carrier of B_3.

One can prove the following propositions:

(36) $(\top_{(B_3)})^{\mathrm{c}} = \bot_{(B_3)}$.

(37) $(\bot_{(B_3)})^{\mathrm{c}} = \top_{(B_3)}$.

(38) If f is epimorphism, then C_1 is Boolean and f preserves complement.

Let us consider B_3. A non empty subset of the carrier of B_3 is called a field of subsets of B_3 if:

(Def.14) If $a \in$ it and $b \in$ it, then $a \sqcap b \in$ it and $a^{\mathrm{c}} \in$ it.

In the sequel F will denote a field of subsets of B_3.

Next we state four propositions:

(39) If $a \in F$ and $b \in F$, then $a \sqcup b \in F$.

(40) If $a \in F$ and $b \in F$, then $a \Rightarrow b \in F$.

(41) The carrier of B_3 is a field of subsets of B_3.

(42) F is a closed subset of B_3.

Let us consider B_3, A. The field by A yielding a field of subsets of B_3 is defined as follows:

(Def.15) $A \subseteq$ the field by A and for every F such that $A \subseteq F$ holds the field by $A \subseteq F$.

Let us consider B_3, A. The functor SetImp(A) yielding a non empty subset of the carrier of B_3 is defined by:

(Def.16) $\mathrm{SetImp}(A) = \{a \Rightarrow b : a \in A \ \wedge \ b \in A\}$.

The following two propositions are true:

(43) $x \in \mathrm{SetImp}(A)$ iff there exist p, q such that $x = p \Rightarrow q$ and $p \in A$ and $q \in A$.

(44) $c \in \mathrm{SetImp}(A)$ iff there exist p, q such that $c = p^{\mathrm{c}} \sqcup q$ and $p \in A$ and $q \in A$.

Let us consider B_3. The functor comp B_3 yielding a function from the carrier of B_3 into the carrier of B_3 is defined by:

(Def.17) $(\mathrm{comp}\, B_3)(a) = a^{\mathrm{c}}$.

We now state several propositions:

(45) $\bigsqcup^{\mathrm{f}}_{B \cup \{b\}} \mathrm{comp}\, B_3 = \bigsqcup^{\mathrm{f}}_{B} \mathrm{comp}\, B_3 \sqcup b^{\mathrm{c}}$.

(46) $(\bigsqcup^{\mathrm{f}}_{B})^{\mathrm{c}} = \bigsqcap^{\mathrm{f}}_{B} \mathrm{comp}\, B_3$.

(47) $\bigsqcap^{\mathrm{f}}_{B \cup \{b\}} \mathrm{comp}\, B_3 = \bigsqcap^{\mathrm{f}}_{B} \mathrm{comp}\, B_3 \sqcap b^{\mathrm{c}}$.

(48) $(\bigsqcap^{\mathrm{f}}_{B})^{\mathrm{c}} = \bigsqcup^{\mathrm{f}}_{B} \mathrm{comp}\, B_3$.

(49) Let A_1 be a closed subset of B_3. Suppose $\bot_{(B_3)} \in A_1$ and $\top_{(B_3)} \in A_1$. Given B. If $B \subseteq \mathrm{SetImp}(A_1)$, then there exists B_0 such that $B_0 \subseteq \mathrm{SetImp}(A_1)$ and $\bigsqcup^{\mathrm{f}}_B \mathrm{comp}\, B_3 = \bigsqcap^{\mathrm{f}}_{(B_0)}$.

(50) For every closed subset A_1 of B_3 such that $\bot_{(B_3)} \in A_1$ and $\top_{(B_3)} \in A_1$ holds $\{\bigsqcap^{\mathrm{f}}_B : B \subseteq \mathrm{SetImp}(A_1)\} =$ the field by A_1.

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Representation Theorem for Heyting Lattices

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The articles [11], [4], [5], [3], [9], [10], [7], [12], [13], [8], [1], [2], and [6] provide the notation and terminology for this paper.

One can check that every lower bound lattice which is Heyting is also implicative and every lattice which is implicative is also upper-bounded.

In the sequel T will denote a topological space and A, B, C will denote subsets of the carrier of T.

We now state two propositions:

(1) $A \cap \mathrm{Int}(A^{\rm c} \cup B) \subseteq B$.

(2) If C is open and $A \cap C \subseteq B$, then $C \subseteq \mathrm{Int}(A^{\rm c} \cup B)$.

Let us consider T. The functor Topology(T) yields a non empty family of subsets of the carrier of T and is defined as follows:

(Def.1) Topology(T) = the topology of T.

In the sequel P, Q denote elements of Topology(T).

The following proposition is true

(3) A is open iff $A \in$ Topology(T).

Let us consider T, P, Q. Then $P \cup Q$ is an element of Topology(T).

Let us consider T, P, Q. Then $P \cap Q$ is an element of Topology(T).

Let us consider T. The functor TopUnion(T) yields a binary operation on Topology(T) and is defined by:

(Def.2) (TopUnion(T))(P, Q) = $P \cup Q$.

Let us consider T. The functor TopMeet(T) yielding a binary operation on Topology(T) is defined as follows:

(Def.3) (TopMeet(T))(P, Q) = $P \cap Q$.

The following proposition is true

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(4) For every topological space T holds $\langle \mathrm{Topology}(T), \mathrm{TopUnion}(T), \mathrm{TopMeet}(T)\rangle$ is a lattice.

Let us consider T. The functor $\mathrm{OpenSetLatt}(T)$ yields a lattice and is defined by:

(Def.4) $\mathrm{OpenSetLatt}(T) = \langle \mathrm{Topology}(T), \mathrm{TopUnion}(T), \mathrm{TopMeet}(T)\rangle$.

Next we state the proposition

(5) The carrier of $\mathrm{OpenSetLatt}(T) = \mathrm{Topology}(T)$.

In the sequel p, q will denote elements of the carrier of $\mathrm{OpenSetLatt}(T)$.

Next we state several propositions:

(6) $p \sqcup q = p \cup q$ and $p \sqcap q = p \cap q$.

(7) $p \sqsubseteq q$ iff $p \subseteq q$.

(8) For all elements p', q' of $\mathrm{Topology}(T)$ such that $p = p'$ and $q = q'$ holds $p \sqsubseteq q$ iff $p' \subseteq q'$.

(9) $\mathrm{OpenSetLatt}(T)$ is implicative.

(10) $\mathrm{OpenSetLatt}(T)$ is lower-bounded and $\perp_{\mathrm{OpenSetLatt}(T)} = \emptyset$.

(11) $\top_{\mathrm{OpenSetLatt}(T)} =$ the carrier of T.

Let us consider T. Then $\mathrm{OpenSetLatt}(T)$ is a Heyting lattice.

For simplicity we adopt the following convention: L will denote a distributive lattice, F will denote a filter of L, a, b will denote elements of the carrier of L, x will be arbitrary, and X_1, X_2, Y, Z will denote sets.

Let us consider L. The functor $\mathrm{PrimeFilters}(L)$ yielding a set is defined as follows:

(Def.5) $\mathrm{PrimeFilters}(L) = \{F : F \neq$ the carrier of $L \ \wedge\ F$ is prime$\}$.

We now state the proposition

(12) $F \in \mathrm{PrimeFilters}(L)$ iff $F \neq$ the carrier of L and F is prime.

Let us consider L. The functor $\mathrm{StoneH}(L)$ yielding a function is defined by:

(Def.6) $\mathrm{dom}\,\mathrm{StoneH}(L) =$ the carrier of L and $(\mathrm{StoneH}(L))(a) = \{F : F \in \mathrm{PrimeFilters}(L) \ \wedge\ a \in F\}$.

Next we state two propositions:

(13) $F \in (\mathrm{StoneH}(L))(a)$ iff $F \in \mathrm{PrimeFilters}(L)$ and $a \in F$.

(14) $x \in (\mathrm{StoneH}(L))(a)$ iff there exists F such that $F = x$ and $F \neq$ the carrier of L and F is prime and $a \in F$.

Let us consider L. The functor $\mathrm{StoneS}(L)$ yielding a non empty set is defined as follows:

(Def.7) $\mathrm{StoneS}(L) = \mathrm{rng}\,\mathrm{StoneH}(L)$.

The following propositions are true:

(15) $x \in \mathrm{StoneS}(L)$ iff there exists a such that $x = (\mathrm{StoneH}(L))(a)$.

(16) $(\mathrm{StoneH}(L))(a \sqcup b) = (\mathrm{StoneH}(L))(a) \cup (\mathrm{StoneH}(L))(b)$.

(17) $(\mathrm{StoneH}(L))(a \sqcap b) = (\mathrm{StoneH}(L))(a) \cap (\mathrm{StoneH}(L))(b)$.

Let us consider L and let us consider a. The functor $\mathrm{Filters}(a)$ yields a non empty family of subsets of L and is defined by:

(Def.8) Filters(a) = $\{F : a \in F\}$.

The following propositions are true:

(18) $x \in$ Filters(a) iff x is a filter of L and $a \in x$.

(19) If $x \in$ Filters(b) $\setminus$ Filters(a), then x is a filter of L and $b \in x$ and $a \notin x$.

(20) Given Z. Suppose $Z \neq \emptyset$ and $Z \subseteq$ Filters(b) $\setminus$ Filters(a) and for all X_1, X_2 such that $X_1 \in Z$ and $X_2 \in Z$ holds $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$. Then there exists Y such that $Y \in$ Filters(b) $\setminus$ Filters(a) and for every X_1 such that $X_1 \in Z$ holds $X_1 \subseteq Y$.

(21) If $b \not\sqsubseteq a$, then $[b) \in$ Filters(b) $\setminus$ Filters(a).

(22) If $b \not\sqsubseteq a$, then there exists F such that $F \in$ PrimeFilters(L) and $a \notin F$ and $b \in F$.

(23) If $a \neq b$, then there exists F such that $F \in$ PrimeFilters(L).

(24) If $a \neq b$, then (StoneH(L))(a) $\neq$ (StoneH(L))(b).

(25) StoneH(L) is one-to-one.

Let us consider L and let A, B be elements of StoneS(L). Then $A \cup B$ is an element of StoneS(L).

Let us consider L and let A, B be elements of StoneS(L). Then $A \cap B$ is an element of StoneS(L).

Let us consider L. The functor SetUnion(L) yielding a binary operation on StoneS(L) is defined as follows:

(Def.9) For all elements A, B of StoneS(L) holds (SetUnion(L))(A, B) = $A \cup B$.

Let us consider L. The functor SetMeet(L) yielding a binary operation on StoneS(L) is defined by:

(Def.10) For all elements A, B of StoneS(L) holds (SetMeet(L))(A, B) = $A \cap B$.

The following proposition is true

(26) $\langle$StoneS(L), SetUnion(L), SetMeet(L)$\rangle$ is a lattice.

Let us consider L. The functor StoneLatt(L) yields a lattice and is defined by:

(Def.11) StoneLatt(L) = $\langle$StoneS(L), SetUnion(L), SetMeet(L)$\rangle$.

In the sequel p, q are elements of the carrier of StoneLatt(L).

We now state three propositions:

(27) For every L holds the carrier of StoneLatt(L) = StoneS(L).

(28) $p \sqcup q = p \cup q$ and $p \sqcap q = p \cap q$.

(29) $p \sqsubseteq q$ iff $p \subseteq q$.

Let us consider L. Then StoneH(L) is a homomorphism from L to StoneLatt(L).

One can prove the following propositions:

(30) StoneH(L) is isomorphism.

(31) StoneLatt(L) is distributive.

(32) L and StoneLatt(L) are isomorphic.

Let us note that there exists a Heyting lattice which is non trivial.

In the sequel H denotes a non trivial Heyting lattice and p', q' denote elements of the carrier of H.

The following three propositions are true:

(33) $(\mathrm{StoneH}(H))(\top_H) = \mathrm{PrimeFilters}(H)$.

(34) $(\mathrm{StoneH}(H))(\bot_H) = \emptyset$.

(35) $\mathrm{StoneS}(H) \subseteq 2^{\mathrm{PrimeFilters}(H)}$.

Let us consider H. Then PrimeFilters(H) is a non empty set.

Let us consider H. The functor HTopSpace(H) yielding a strict topological space is defined as follows:

(Def.12) The carrier of HTopSpace(H) = PrimeFilters(H) and the topology of HTopSpace(H) = $\{\bigcup A : A$ ranges over subsets of StoneS(H), $\}$.

One can prove the following propositions:

(36) The carrier of OpenSetLatt(HTopSpace(H)) = $\{\bigcup A : A$ ranges over subsets of StoneS(H), $\}$.

(37) StoneS(H) $\subseteq$ the carrier of OpenSetLatt(HTopSpace(H)).

Let us consider H. Then StoneH(H) is a homomorphism from H to OpenSetLatt(HTopSpace(H)).

The following propositions are true:

(38) StoneH(H) is monomorphism.

(39) $(\mathrm{StoneH}(H))(p' \Rightarrow q') = (\mathrm{StoneH}(H))(p') \Rightarrow (\mathrm{StoneH}(H))(q')$.

(40) StoneH(H) preserves implication.

(41) StoneH(H) preserves top.

(42) StoneH(H) preserves bottom.

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Representation Theorem for Boolean Algebras

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The notation and terminology used in this paper are introduced in the following articles: [9], [7], [4], [5], [3], [10], [11], [8], [12], [1], [2], and [6].

In the sequel T is a topological space, X, Y are subsets of T, and x is arbitrary.

Let T be a topological space. The functor OpenClosedSet(T) yielding a non empty family of subsets of the carrier of T is defined as follows:

(Def.1) OpenClosedSet(T) $= \{x : x$ ranges over subsets of T, x is open $\land$ x is closed$\}$.

The following propositions are true:

(1) If $x \in$ OpenClosedSet(T), then there exists X such that $X = x$.

(2) If $X \in$ OpenClosedSet(T), then X is open.

(3) If $X \in$ OpenClosedSet(T), then X is closed.

(4) If X is open and closed, then $X \in$ OpenClosedSet(T).

Let X be a non empty set and let t be a non empty family of subsets of X. We see that the element of t is a subset of X.

In the sequel x, y, z will denote elements of OpenClosedSet(T).

Let us consider T and let C, D be elements of OpenClosedSet(T). Then $C \cup D$ is an element of OpenClosedSet(T).

Let us consider T and let C, D be elements of OpenClosedSet(T). Then $C \cap D$ is an element of OpenClosedSet(T).

Let us consider T. The functor join(T) yielding a binary operation on OpenClosedSet(T) is defined by:

(Def.2) For all elements A, B of OpenClosedSet(T) holds $(\mathrm{join}(T))(A, B) = A \cup B$.

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Let us consider T. The functor meet(T) yields a binary operation on OpenClosedSet(T) and is defined by:

(Def.3) For all elements A, B of OpenClosedSet(T) holds $(\mathrm{meet}(T))(A, B) = A \cap B$.

We now state several propositions:

(5) Let x, y be elements of the carrier of $\langle$OpenClosedSet(T), join(T), meet(T)$\rangle$ and let x', y' be elements of OpenClosedSet(T). If $x = x'$ and $y = y'$, then $x \sqcup y = x' \cup y'$.

(6) Let x, y be elements of the carrier of $\langle$OpenClosedSet(T), join(T), meet(T)$\rangle$ and let x', y' be elements of OpenClosedSet(T). If $x = x'$ and $y = y'$, then $x \sqcap y = x' \cap y'$.

(7) $\emptyset_T$ is an element of OpenClosedSet(T).

(8) Ω_T is an element of OpenClosedSet(T).

(9) For every element x of OpenClosedSet(T) holds x^{c} is an element of OpenClosedSet(T).

(10) $\langle$OpenClosedSet(T), join(T), meet(T)$\rangle$ is a lattice.

Let T be a topological space. The functor OpenClosedSetLatt(T) yields a lattice and is defined by:

(Def.4) OpenClosedSetLatt(T) = $\langle$OpenClosedSet(T), join(T), meet(T)$\rangle$.

Next we state two propositions:

(11) For every topological space T and for all elements x, y of the carrier of OpenClosedSetLatt(T) holds $x \sqcup y = x \cup y$.

(12) For every topological space T and for all elements x, y of the carrier of OpenClosedSetLatt(T) holds $x \sqcap y = x \cap y$.

We follow a convention: a, b, c denote elements of the carrier of $\langle$OpenClosedSet(T), join(T), meet(T)$\rangle$ and x, y, z denote elements of OpenClosedSet(T).

The following propositions are true:

(13) The carrier of OpenClosedSetLatt(T) = OpenClosedSet(T).

(14) OpenClosedSetLatt(T) is Boolean.

(15) Ω_T is an element of the carrier of OpenClosedSetLatt(T).

(16) $\emptyset_T$ is an element of the carrier of OpenClosedSetLatt(T).

One can check that there exists a Boolean lattice which is non trivial.

For simplicity we adopt the following convention: L_1, L_2 denote lattices, a, p, q' denote elements of the carrier of B_1, U_1 denotes a filter of B_1, B denotes a subset of the carrier of B_1, and D denotes a non empty subset of the carrier of B_1.

Let us consider B_1. The functor ultraset(B_1) yields a non empty subset of $2^{\text{the carrier of } B_1}$ and is defined by:

(Def.5) ultraset(B_1) = $\{F : F \text{ is ultrafilter}\}$.

Next we state two propositions:

(18)[1] $x \in$ ultraset(B_1) iff there exists U_1 such that $U_1 = x$ and U_1 is ultrafilter.

(19) For every a holds $\{F : F$ is ultrafilter $\wedge\ a \in F\} \subseteq$ ultraset(B_1).

Let us consider B_1. The functor UFilter(B_1) yielding a function is defined as follows:

(Def.6) dom UFilter(B_1) = the carrier of B_1 and for every element a of the carrier of B_1 holds (UFilter(B_1))(a) = $\{U_1 : U_1$ is ultrafilter $\wedge\ a \in U_1\}$.

Next we state several propositions:

(20) $x \in$ (UFilter(B_1))(a) iff there exists F such that $F = x$ and F is ultrafilter and $a \in F$.

(21) $F \in$ (UFilter(B_1))(a) iff F is ultrafilter and $a \in F$.

(22) For every F such that F is ultrafilter holds $a \sqcup b \in F$ iff $a \in F$ or $b \in F$.

(23) (UFilter(B_1))($a \sqcap b$) = (UFilter(B_1))(a) $\cap$ (UFilter(B_1))(b).

(24) (UFilter(B_1))($a \sqcup b$) = (UFilter(B_1))(a) $\cup$ (UFilter(B_1))(b).

Let us consider B_1. Then UFilter(B_1) is a function from the carrier of B_1 into $2^{\mathrm{ultraset}(B_1)}$.

Let us consider B_1. The functor StoneR(B_1) yielding a non empty set is defined as follows:

(Def.7) StoneR(B_1) = rng UFilter(B_1).

The following propositions are true:

(25) StoneR(B_1) $\subseteq 2^{\mathrm{ultraset}(B_1)}$.

(26) $x \in$ StoneR(B_1) iff there exists a such that (UFilter(B_1))(a) = x.

Let us consider B_1. The functor StoneSpace(B_1) yielding a strict topological space is defined by:

(Def.8) The carrier of StoneSpace(B_1) = ultraset(B_1) and the topology of StoneSpace(B_1) = $\{\bigcup A : A$ ranges over subsets of $2^{\mathrm{ultraset}(B_1)}$, $A \subseteq$ StoneR(B_1)$\}$.

One can prove the following two propositions:

(27) If F is ultrafilter and $F \notin$ (UFilter(B_1))(a), then $a \notin F$.

(28) ultraset(B_1) $\setminus$ (UFilter(B_1))(a) = (UFilter(B_1))(a^{c}).

Let us consider B_1. The functor StoneBLattice(B_1) yields a lattice and is defined as follows:

(Def.9) StoneBLattice(B_1) = OpenClosedSetLatt(StoneSpace(B_1)).

One can prove the following four propositions:

(29) UFilter(B_1) is one-to-one.

(30) $\bigcup$ StoneR(B_1) = ultraset(B_1).

(31) For all sets A, B, X such that $X \subseteq \bigcup(A \cup B)$ and for arbitrary Y such that $Y \in B$ holds $Y \cap X = \emptyset$ holds $X \subseteq \bigcup A$.

(32) For every non empty set X holds there exists finite subset of X which is non empty.

[1]The proposition (17) has been removed.

Let D be a non empty set. Note that there exists a finite subset of D which is non empty.

The following propositions are true:

(33) For every lattice L and for all elements a, b, c, d of the carrier of L such that $a \sqsubseteq c$ and $b \sqsubseteq d$ holds $a \sqcap b \sqsubseteq c \sqcap d$.

(34) Let L be a non trivial Boolean lattice and let D be a non empty subset of the carrier of L. Suppose $\bot_L \in [D)$. Then there exists a non empty finite subset B of the carrier of L such that $B \subseteq D$ and $\bigsqcap^{\rm f}_B = \bot_L$.

(35) For every lower bound lattice L it is not true that there exists a filter F of L such that F is ultrafilter and $\bot_L \in F$.

(36) $(\mathrm{UFilter}(B_1))(\bot_{(B_1)}) = \emptyset$.

(37) $(\mathrm{UFilter}(B_1))(\top_{(B_1)}) = \mathrm{ultraset}(B_1)$.

(38) If $\mathrm{ultraset}(B_1) = \bigcup X$ and X is a subset of $\mathrm{StoneR}(B_1)$, then there exists a finite subset Y of X such that $\mathrm{ultraset}(B_1) = \bigcup Y$.

(39) If $x \in 2^X$ and $y \in 2^X$, then $x \cap y \in 2^X$.

(40) $\mathrm{StoneR}(B_1) = \mathrm{OpenClosedSet}(\mathrm{StoneSpace}(B_1))$.

Let us consider B_1. Then $\mathrm{UFilter}(B_1)$ is a homomorphism from B_1 to $\mathrm{StoneBLattice}(B_1)$.

Next we state four propositions:

(41) $\mathrm{rng}\,\mathrm{UFilter}(B_1) =$ the carrier of $\mathrm{StoneBLattice}(B_1)$.

(42) $\mathrm{UFilter}(B_1)$ is isomorphism.

(43) B_1 and $\mathrm{StoneBLattice}(B_1)$ are isomorphic.

(44) For every non trivial Boolean lattice B_1 there exists a topological space T such that B_1 and $\mathrm{OpenClosedSetLatt}(T)$ are isomorphic.

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Some Remarks on the Simple Concrete Model of Computer

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Summary. We prove some results on **SCM** needed for the proof of the correctness of Euclid's algorithm. We introduce the following concepts:

- starting finite partial state (Start-At(l)), then assigns to the instruction counter an instruction location (and consists only of this assignment),
- programmed finite partial state, that consists of the instructions (to be more precise, a finite partial state with the domain consisting of instruction locations).

We define for a total state s what it means that s starts at l (the value of the instruction counter in the state s is l) and s halts at l (the halt instruction is assigned to l in the state s). Similar notions are defined for finite partial states.

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The articles [22], [20], [5], [6], [21], [12], [1], [17], [23], [4], [13], [2], [18], [24], [7], [19], [8], [9], [11], [3], [10], [14], [15], and [16] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following proposition

(1) For all integers m, j holds $m \cdot j \equiv +0 (\mathrm{mod}\, m)$.

In the sequel i, j, k will denote natural numbers.

The scheme *INDI* concerns natural numbers $\mathcal{A}$, $\mathcal{B}$ and a unary predicate $\mathcal{P}$, and states that:

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$\mathcal{P}[\mathcal{B}]$

provided the following requirements are met:

- $\mathcal{P}[0]$,
- $\mathcal{A} > 0$,
- For all i, j such that $\mathcal{P}[\mathcal{A} \cdot i]$ and $j \neq 0$ and $j \leq \mathcal{A}$ holds $\mathcal{P}[\mathcal{A} \cdot i + j]$.

In the sequel x will be arbitrary.

Next we state a number of propositions:

(2) Let X, Y be non empty set and let f, g be partial functions from X to Y. Suppose that for every element x of X and for every element y of Y holds $\langle x, y\rangle \in f$ iff $\langle x, y\rangle \in g$. Then $f = g$.

(3) For all functions f, g and for all sets A, B such that $f \upharpoonright A = g \upharpoonright A$ and $f \upharpoonright B = g \upharpoonright B$ holds $f \upharpoonright (A \cup B) = g \upharpoonright (A \cup B)$.

(4) For every set X and for all functions f, g such that $\operatorname{dom} g \subseteq X$ and $g \subseteq f$ holds $g \subseteq f \upharpoonright X$.

(5) For every function f and for arbitrary x such that $x \in \operatorname{dom} f$ holds $f \upharpoonright \{x\} = \{\langle x, f(x)\rangle\}$.

(6) For every function f and for every set X such that $X \cap \operatorname{dom} f = \emptyset$ holds $f \upharpoonright X = \emptyset$.

(7) For all functions f, g and for arbitrary x such that $\operatorname{dom} f = \operatorname{dom} g$ and $f(x) = g(x)$ holds $f \upharpoonright \{x\} = g \upharpoonright \{x\}$.

(8) For all functions f, g and for arbitrary x, y such that $\operatorname{dom} f = \operatorname{dom} g$ and $f(x) = g(x)$ and $f(y) = g(y)$ holds $f \upharpoonright \{x, y\} = g \upharpoonright \{x, y\}$.

(9) Let f, g be functions and let x, y, z be arbitrary. If $\operatorname{dom} f = \operatorname{dom} g$ and $f(x) = g(x)$ and $f(y) = g(y)$ and $f(z) = g(z)$, then $f \upharpoonright \{x, y, z\} = g \upharpoonright \{x, y, z\}$.

(10) For arbitrary a, b and for every function f such that $a \in \operatorname{dom} f$ and $f(a) = b$ holds $a \dot{\longmapsto} b \subseteq f$.

(11) For arbitrary a, b, c, d such that $a \neq c$ holds $[a \longmapsto b, c \longmapsto d] = \{\langle a, b\rangle, \langle c, d\rangle\}$.

(12) For arbitrary a, b, c, d and for every function f such that $a \in \operatorname{dom} f$ and $c \in \operatorname{dom} f$ and $f(a) = b$ and $f(c) = d$ holds $[a \longmapsto b, c \longmapsto d] \subseteq f$.

(13) For all functions f, g, h holds $(f +\cdot g) +\cdot h = f +\cdot (g +\cdot h)$.

2. Computations

In the sequel N denotes a non empty set with non empty elements.

Next we state the proposition

(14) For every AMI S over N and for every finite partial state p of S holds $p \in \operatorname{FinPartSt}(S)$.

Let us consider N and let S be an AMI over N. Then $\operatorname{FinPartSt}(S)$ is a non empty subset of $\prod^{\cdot}$ (the object kind of S).

Next we state two propositions:

(15) For every AMI S over N holds every element of FinPartSt(S) is a finite partial state of S.

(16) Let S be an AMI over N and let F_1, F_2 be partial functions from FinPartSt(S) to FinPartSt(S). Suppose that for all finite partial states p, q of S holds $\langle p, q\rangle \in F_1$ iff $\langle p, q\rangle \in F_2$. Then $F_1 = F_2$.

The scheme *EqFPSFunc* concerns a non empty set $\mathcal{A}$ with non empty elements, an AMI $\mathcal{B}$ over $\mathcal{A}$, partial functions $\mathcal{C}$, $\mathcal{D}$ from FinPartSt($\mathcal{B}$) to FinPartSt($\mathcal{B}$), and a binary predicate $\mathcal{P}$, and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters meet the following conditions:

- For all finite partial states p, q of $\mathcal{B}$ holds $\langle p, q\rangle \in \mathcal{C}$ iff $\mathcal{P}[p, q]$,
- For all finite partial states p, q of $\mathcal{B}$ holds $\langle p, q\rangle \in \mathcal{D}$ iff $\mathcal{P}[p, q]$.

Let us consider N, let S be a von Neumann definite AMI over N, and let l be an instruction-location of S. The functor Start-At(l) yielding a finite partial state of S is defined by:

(Def.1) Start-At(l) = $\mathbf{IC}_S \longmapsto l$.

One can prove the following proposition

(17) For every von Neumann definite AMI S over N and for every instruction-location l of S holds dom Start-At(l) = $\{\mathbf{IC}_S\}$.

Let us consider N and let S be an AMI over N. A finite partial state of S is programmed if:

(Def.2) dom it $\subseteq$ the instruction locations of S.

We now state four propositions:

(18) Let S be a steady-programmed von Neumann definite AMI over N and let p_1, p_2 be programmed finite partial state of S. Then $p_1 +\cdot\, p_2$ is programmed.

(19) For every AMI S over N and for every state s of S holds dom s = the objects of S.

(20) For every AMI S over N and for every finite partial state p of S holds dom $p \subseteq$ the objects of S.

(21) Let S be a steady-programmed von Neumann definite AMI over N, and let p be a programmed finite partial state of S, and let s be a state of S. If $p \subseteq s$, then for every k holds $p \subseteq (\text{Computation}(s))(k)$.

Let us consider N, let S be a von Neumann AMI over N, let s be a state of S, and let l be an instruction-location of S. We say that s starts at l if and only if:

(Def.3) $\mathbf{IC}_s = l$.

We say that s halts at l if and only if:

(Def.4) $s(l) = \mathbf{halt}_S$.

The following proposition is true

(22) For every AMI S over N and for every finite partial state p of S there exists a state s of S such that $p \subseteq s$.

Let us consider N, let S be a definite von Neumann AMI over N, and let p be a finite partial state of S. Let us assume that $\mathbf{IC}_S \in \operatorname{dom} p$. The functor $\mathbf{IC}_p$ yielding an instruction-location of S is defined by:

(Def.5) $\mathbf{IC}_p = p(\mathbf{IC}_S)$.

Let us consider N, let S be a definite von Neumann AMI over N, let p be a finite partial state of S, and let l be an instruction-location of S. We say that p starts at l if and only if:

(Def.6) $\mathbf{IC}_S \in \operatorname{dom} p$ and $\mathbf{IC}_p = l$.

We say that p halts at l if and only if:

(Def.7) $l \in \operatorname{dom} p$ and $p(l) = \mathbf{halt}_S$.

One can prove the following propositions:

(23) Let S be a von Neumann definite steady-programmed AMI over N and let s be a state of S. Then s is halting if and only if there exists k such that s halts at $\mathbf{IC}_{(\mathrm{Computation}(s))(k)}$.

(24) Let S be a von Neumann definite steady-programmed AMI over N, and let s be a state of S, and let p be a finite partial state of S, and let l be an instruction-location of S. If $p \subseteq s$ and p halts at l, then s halts at l.

(25) Let S be a halting steady-programmed von Neumann definite AMI over N, and let s be a state of S, and given k. If s is halting, then $\mathrm{Result}(s) = (\mathrm{Computation}(s))(k)$ iff s halts at $\mathbf{IC}_{(\mathrm{Computation}(s))(k)}$.

(26) Let S be a steady-programmed von Neumann definite AMI over N, and let s be a state of S, and let p be a programmed finite partial state of S, and given k. Then $p \subseteq s$ if and only if $p \subseteq (\mathrm{Computation}(s))(k)$.

(27) Let S be a halting steady-programmed von Neumann definite AMI over N, and let s be a state of S, and given k. If s halts at $\mathbf{IC}_{(\mathrm{Computation}(s))(k)}$, then $\mathrm{Result}(s) = (\mathrm{Computation}(s))(k)$.

(28) Suppose $i \leq j$. Let S be a halting steady-programmed von Neumann definite AMI over N and let s be a state of S. If s halts at $\mathbf{IC}_{(\mathrm{Computation}(s))(i)}$, then s halts at $\mathbf{IC}_{(\mathrm{Computation}(s))(j)}$.

(29) Suppose $i \leq j$. Let S be a halting steady-programmed von Neumann definite AMI over N and let s be a state of S. If s halts at $\mathbf{IC}_{(\mathrm{Computation}(s))(i)}$, then $(\mathrm{Computation}(s))(j) = (\mathrm{Computation}(s))(i)$.

(30) Let S be a steady-programmed von Neumann halting definite AMI over N and let s be a state of S. If there exists k such that s halts at $\mathbf{IC}_{(\mathrm{Computation}(s))(k)}$, then for every i holds $\mathrm{Result}(s) = \mathrm{Result}((\mathrm{Computation}(s))(i))$.

(31) Let S be a steady-programmed von Neumann definite AMI over N, and let s be a state of S, and let l be an instruction-location of S, and given k. Then s halts at l if and only if $(\mathrm{Computation}(s))(k)$ halts at l.

(32) Let S be a definite von Neumann AMI over N, and let p be a finite partial state of S, and let l be an instruction-location of S. Suppose p starts at l. Let s be a state of S. If $p \subseteq s$, then s starts at l.

(33) For every von Neumann definite AMI S over N and for every instruction-location l of S holds $\text{Start-At}(l)(\mathbf{IC}_S) = l$.

Let us consider N, let S be a definite von Neumann AMI over N, let l be an instruction-location of S, and let I be an instruction of S. Then $l \mapsto I$ is a programmed finite partial state of S.

3. Instruction Locations and Data Locations

We now state the proposition

(34) **SCM** is realistic.

SCM is a steady-programmed halting realistic von Neumann data-oriented definite strict AMI over $\{\mathbb{Z}\}$.

Let us consider k. The functor $\mathbf{d}_k$ yields a data-location and is defined by:

(Def.8) $\mathbf{d}_k = 2 \cdot k + 1$.

The functor $\mathbf{i}_k$ yielding an instruction-location of **SCM** is defined by:

(Def.9) $\mathbf{i}_k = 2 \cdot k + 2$.

Next we state three propositions:

(35) For all i, j such that $i \neq j$ holds $\mathbf{d}_i \neq \mathbf{d}_j$.

(36) For all i, j such that $i \neq j$ holds $\mathbf{i}_i \neq \mathbf{i}_j$.

(37) $\text{Next}(\mathbf{i}_k) = \mathbf{i}_{k+1}$.

Let s be a state of **SCM** and let a be a data-location. Then $s(a)$ is an integer.

Let us consider a, b. Then $a{:=}b$ is an instruction of **SCM**. Then $\text{AddTo}(a, b)$ is an instruction of **SCM**. Then $\text{SubFrom}(a, b)$ is an instruction of **SCM**. Then $\text{MultBy}(a, b)$ is an instruction of **SCM**. Then $\text{Divide}(a, b)$ is an instruction of **SCM**.

Let us consider l_1. Then goto l_1 is an instruction of **SCM**. Let us consider a. Then **if** $a = 0$ **goto** l_1 is an instruction of **SCM**. Then **if** $a > 0$ **goto** l_1 is an instruction of **SCM**.

Next we state the proposition

(38) For every data-location l holds $\text{ObjectKind}(l) = \mathbb{Z}$.

Let l_2 be a data-location and let a be an integer. Then $l_2 \mapsto a$ is a finite partial state of **SCM**.

Let l_2, l_3 be data-locations and let a, b be integers. Then $[l_2 \longmapsto a, l_3 \longmapsto b]$ is a finite partial state of **SCM**.

Next we state two propositions:

(39) For all i, j holds $\mathbf{d}_i \neq \mathbf{i}_j$.

(40) For every i holds $\mathbf{IC}_{\mathbf{SCM}} \neq \mathbf{d}_i$ and $\mathbf{IC}_{\mathbf{SCM}} \neq \mathbf{i}_i$.

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Euclid's Algorithm

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Summary. The main goal of the paper is to prove the correctness of the Euclid's algorithm for **SCM**. We define the Euclid's algorithm and describe the natural semantics of it. Eventually we prove that the Euclid's algorithm computes the Euclid's function. Let us observe that the Euclid's function is defined as a function mapping finite partial states to finite partial states of **SCM** rather than pairs of integers to integers.

MML Identifier: AMI_4.

The papers [20], [18], [5], [6], [19], [11], [1], [15], [22], [4], [12], [2], [16], [23], [17], [7], [8], [10], [3], [9], [13], [14], and [21] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(1) For all integers i, j such that $i \geq 0$ and $j > 0$ holds $i \div j \geq 0$.

(2) For all integers i, j such that $i \geq 0$ and $j > 0$ holds $|i| \bmod |j| = i \bmod j$ and $|i| \div |j| = i \div j$.

In the sequel i, j, k denote natural numbers.

Next we state the proposition

(3) For all i, j such that $i > 0$ and $j > 0$ holds $\gcd(i,j) > 0$.

The scheme *Euklides'* concerns a unary functor $\mathcal{F}$ yielding a natural number, a unary functor $\mathcal{G}$ yielding a natural number, a natural number $\mathcal{A}$, and a natural number $\mathcal{B}$, and states that:

There exists k such that $\mathcal{F}(k) = \gcd(\mathcal{A}, \mathcal{B})$ and $\mathcal{G}(k) = 0$

provided the following requirements are met:

- $0 < \mathcal{B}$,

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- $\mathcal{B} < \mathcal{A}$,
- $\mathcal{F}(0) = \mathcal{A}$,
- $\mathcal{G}(0) = \mathcal{B}$,
- For every k such that $\mathcal{G}(k) > 0$ holds $\mathcal{F}(k+1) = \mathcal{G}(k)$ and $\mathcal{G}(k+1) = \mathcal{F}(k) \bmod \mathcal{G}(k)$.

2. Euclid's Algorithm

The Euclid's algorithm is a programmed finite partial state of **SCM** and is defined by:

(Def.1) The Euclid's algorithm $= (\mathbf{i}_0 \dot{\longmapsto} (\mathbf{d}_2 := \mathbf{d}_1)) +\cdot ((\mathbf{i}_1 \dot{\longmapsto} \mathrm{Divide}(\mathbf{d}_0, \mathbf{d}_1)) +\cdot ((\mathbf{i}_2 \dot{\longmapsto} (\mathbf{d}_0 := \mathbf{d}_2)) +\cdot ((\mathbf{i}_3 \dot{\longmapsto} (\mathbf{if}\ \mathbf{d}_1 > 0\ \mathbf{goto}\ \mathbf{i}_0)) +\cdot (\mathbf{i}_4 \dot{\longmapsto} \mathbf{halt}_{\mathbf{SCM}}))))$.

Next we state the proposition

(4) $\mathrm{dom}\,(\text{the Euclid's algorithm}) = \{\mathbf{i}_0, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3, \mathbf{i}_4\}$.

3. The Natural Semantics of the Euclid's Algorithm

We now state several propositions:

(5) Let s be a state of **SCM**. Suppose the Euclid's algorithm $\subseteq s$. Given k. Suppose $\mathbf{IC}_{(\mathrm{Computation}(s))(k)} = \mathbf{i}_0$. Then $\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)} = \mathbf{i}_1$ and $(\mathrm{Computation}(s))(k+1)(\mathbf{d}_0) = (\mathrm{Computation}(s))(k)(\mathbf{d}_0)$ and $(\mathrm{Computation}(s))(k+1)(\mathbf{d}_1) = (\mathrm{Computation}(s))(k)(\mathbf{d}_1)$ and $(\mathrm{Computation}(s))(k+1)(\mathbf{d}_2) = (\mathrm{Computation}(s))(k)(\mathbf{d}_1)$.

(6) Let s be a state of **SCM**. Suppose the Euclid's algorithm $\subseteq s$. Given k. Suppose $\mathbf{IC}_{(\mathrm{Computation}(s))(k)} = \mathbf{i}_1$. Then $\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)} = \mathbf{i}_2$ and $(\mathrm{Computation}(s))(k+1)(\mathbf{d}_0) = (\mathrm{Computation}(s))(k)(\mathbf{d}_0) \div (\mathrm{Computation}(s))(k)(\mathbf{d}_1)$ and $(\mathrm{Computation}(s))(k+1)(\mathbf{d}_1) = (\mathrm{Computation}(s))(k)(\mathbf{d}_0) \bmod (\mathrm{Computation}(s))(k)(\mathbf{d}_1)$ and $(\mathrm{Computation}(s))(k+1)(\mathbf{d}_2) = (\mathrm{Computation}(s))(k)(\mathbf{d}_2)$.

(7) Let s be a state of **SCM**. Suppose the Euclid's algorithm $\subseteq s$. Given k. Suppose $\mathbf{IC}_{(\mathrm{Computation}(s))(k)} = \mathbf{i}_2$. Then $\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)} = \mathbf{i}_3$ and $(\mathrm{Computation}(s))(k+1)(\mathbf{d}_0) = (\mathrm{Computation}(s))(k)(\mathbf{d}_2)$ and $(\mathrm{Computation}(s))(k+1)(\mathbf{d}_1) = (\mathrm{Computation}(s))(k)(\mathbf{d}_1)$ and $(\mathrm{Computation}(s))(k+1)(\mathbf{d}_2) = (\mathrm{Computation}(s))(k)(\mathbf{d}_2)$.

(8) Let s be a state of **SCM**. Suppose the Euclid's algorithm $\subseteq s$. Given k. Suppose $\mathbf{IC}_{(\mathrm{Computation}(s))(k)} = \mathbf{i}_3$. Then

(i) if $(\mathrm{Computation}(s))(k)(\mathbf{d}_1) > 0$, then $\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)} = \mathbf{i}_0$,

(ii) if $(\mathrm{Computation}(s))(k)(\mathbf{d}_1) \leq 0$, then $\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)} = \mathbf{i}_4$,

(iii) $(\mathrm{Computation}(s))(k+1)(\mathbf{d}_0) = (\mathrm{Computation}(s))(k)(\mathbf{d}_0)$, and

(iv) $(\mathrm{Computation}(s))(k+1)(\mathbf{d}_1) = (\mathrm{Computation}(s))(k)(\mathbf{d}_1)$.

(9) For every state s of $\mathbf{SCM}$ such that the Euclid's algorithm $\subseteq s$ and for all k, i such that $\mathbf{IC}_{(\mathrm{Computation}(s))(k)} = \mathbf{i}_4$ holds $(\mathrm{Computation}(s))(k+i) = (\mathrm{Computation}(s))(k)$.

(10) Let s be a state of $\mathbf{SCM}$. Suppose s starts at $\mathbf{i}_0$ and the Euclid's algorithm $\subseteq s$. Let x, y be integers. If $s(\mathbf{d}_0) = x$ and $s(\mathbf{d}_1) = y$ and $x > 0$ and $y > 0$, then $(\mathrm{Result}(s))(\mathbf{d}_0) = \gcd(x, y)$.

The Euclid's function is a partial function from FinPartSt($\mathbf{SCM}$) to FinPartSt($\mathbf{SCM}$) and is defined by the condition (Def.2).

(Def.2) Let p, q be finite partial states of $\mathbf{SCM}$. Then $\langle p, q \rangle \in$ the Euclid's function if and only if there exist integers x, y such that $x > 0$ and $y > 0$ and $p = [\mathbf{d}_0 \longmapsto x, \mathbf{d}_1 \longmapsto y]$ and $q = \mathbf{d}_0 \stackrel{\cdot}{\longmapsto} \gcd(x, y)$.

The following three propositions are true:

(11) Let p be arbitrary. Then $p \in \mathrm{dom}$ (the Euclid's function) if and only if there exist integers x, y such that $x > 0$ and $y > 0$ and $p = [\mathbf{d}_0 \longmapsto x, \mathbf{d}_1 \longmapsto y]$.

(12) For all integers i, j such that $i > 0$ and $j > 0$ holds (the Euclid's function)$([\mathbf{d}_0 \longmapsto i, \mathbf{d}_1 \longmapsto j]) = \mathbf{d}_0 \stackrel{\cdot}{\longmapsto} \gcd(i, j)$.

(13) Start-At($\mathbf{i}_0$) $+\!\cdot$ (the Euclid's algorithm) computes the Euclid's function.

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Development of Terminology for SCM [1]

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Summary. We develop a higher level terminology for the **SCM** machine defined by Nakamura and Trybulec in [6]. Among numerous technical definitions and lemmas we define a complexity measure of a halting state of **SCM** and a loader for **SCM** for arbitrary finite sequence of instructions. In order to test the introduced terminology we discuss properties of eight shortest halting programs, one for each instruction.

MML Identifier: SCM_1.

The notation and terminology used in this paper have been introduced in the following articles: [10], [1], [13], [11], [9], [4], [5], [2], [3], [8], [6], [7], and [12].

Let i be an integer. Then $\langle i \rangle$ is a finite sequence of elements of $\mathbb{Z}$.

One can prove the following propositions:

(1) For every state s of **SCM** holds $\mathbf{IC}_s = s(0)$ and $\mathrm{CurInstr}(s) = s(s(0))$.

(2) For every state s of **SCM** and for every natural number k holds $\mathrm{CurInstr}((\mathrm{Computation}(s))(k)) = s(\mathbf{IC}_{(\mathrm{Computation}(s))(k)})$ and $\mathrm{CurInstr}((\mathrm{Computation}(s))(k)) = s((\mathrm{Computation}(s))(k)(0))$.

(3) For every state s of **SCM** such that there exists a natural number k such that $s(\mathbf{IC}_{(\mathrm{Computation}(s))(k)}) = \mathbf{halt}_{\mathbf{SCM}}$ holds s is halting.

(4) For every state s of **SCM** and for every natural number k such that $s(\mathbf{IC}_{(\mathrm{Computation}(s))(k)}) = \mathbf{halt}_{\mathbf{SCM}}$ holds $\mathrm{Result}(s) = (\mathrm{Computation}(s))(k)$.

(5) For all natural numbers k, l such that $k \neq l$ holds $\mathbf{d}_k \neq \mathbf{d}_l$.

(6) For all natural numbers k, l such that $k \neq l$ holds $\mathbf{i}_k \neq \mathbf{i}_l$.

(7) For all natural numbers n, m holds $\mathbf{IC}_{\mathbf{SCM}} \neq \mathbf{i}_n$ and $\mathbf{IC}_{\mathbf{SCM}} \neq \mathbf{d}_n$ and $\mathbf{i}_n \neq \mathbf{d}_m$.

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Let I be a finite sequence of elements of the instructions of **SCM**, let D be a finite sequence of elements of $\mathbb{Z}$, and let i_1, p_1, d_1 be natural numbers. A state of **SCM** is said to be a state with instruction counter on i_1, with I located from p_1, and D from d_1 if it satisfies the conditions (Def.1).

(Def.1) (i) $\mathbf{IC}_{\rm it} = \mathbf{i}_{(i_1)}$,

(ii) for every natural number k such that $k < \operatorname{len} I$ holds $\operatorname{it}(\mathbf{i}_{p_1+k}) = I(k+1)$, and

(iii) for every natural number k such that $k < \operatorname{len} D$ holds $\operatorname{it}(\mathbf{d}_{d_1+k}) = D(k+1)$.

One can prove the following propositions:

(8) Let x_1, x_2, x_3, x_4 be arbitrary and let p be a finite sequence. If $p = \langle x_1 \rangle \frown \langle x_2 \rangle \frown \langle x_3 \rangle \frown \langle x_4 \rangle$, then $\operatorname{len} p = 4$ and $p(1) = x_1$ and $p(2) = x_2$ and $p(3) = x_3$ and $p(4) = x_4$.

(9) Let x_1, x_2, x_3, x_4, x_5 be arbitrary and let p be a finite sequence. Suppose $p = \langle x_1 \rangle \frown \langle x_2 \rangle \frown \langle x_3 \rangle \frown \langle x_4 \rangle \frown \langle x_5 \rangle$. Then $\operatorname{len} p = 5$ and $p(1) = x_1$ and $p(2) = x_2$ and $p(3) = x_3$ and $p(4) = x_4$ and $p(5) = x_5$.

(10) Let x_1, x_2, x_3, x_4, x_5, x_6 be arbitrary and let p be a finite sequence. Suppose $p = \langle x_1 \rangle \frown \langle x_2 \rangle \frown \langle x_3 \rangle \frown \langle x_4 \rangle \frown \langle x_5 \rangle \frown \langle x_6 \rangle$. Then $\operatorname{len} p = 6$ and $p(1) = x_1$ and $p(2) = x_2$ and $p(3) = x_3$ and $p(4) = x_4$ and $p(5) = x_5$ and $p(6) = x_6$.

(11) Let $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ be arbitrary and let p be a finite sequence. Suppose $p = \langle x_1 \rangle \frown \langle x_2 \rangle \frown \langle x_3 \rangle \frown \langle x_4 \rangle \frown \langle x_5 \rangle \frown \langle x_6 \rangle \frown \langle x_7 \rangle$. Then $\operatorname{len} p = 7$ and $p(1) = x_1$ and $p(2) = x_2$ and $p(3) = x_3$ and $p(4) = x_4$ and $p(5) = x_5$ and $p(6) = x_6$ and $p(7) = x_7$.

(12) Let x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 be arbitrary and let p be a finite sequence. Suppose $p = \langle x_1 \rangle \frown \langle x_2 \rangle \frown \langle x_3 \rangle \frown \langle x_4 \rangle \frown \langle x_5 \rangle \frown \langle x_6 \rangle \frown \langle x_7 \rangle \frown \langle x_8 \rangle$. Then $\operatorname{len} p = 8$ and $p(1) = x_1$ and $p(2) = x_2$ and $p(3) = x_3$ and $p(4) = x_4$ and $p(5) = x_5$ and $p(6) = x_6$ and $p(7) = x_7$ and $p(8) = x_8$.

(13) Let x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 be arbitrary and let p be a finite sequence. Suppose $p = \langle x_1 \rangle \frown \langle x_2 \rangle \frown \langle x_3 \rangle \frown \langle x_4 \rangle \frown \langle x_5 \rangle \frown \langle x_6 \rangle \frown \langle x_7 \rangle \frown \langle x_8 \rangle \frown \langle x_9 \rangle$. Then $\operatorname{len} p = 9$ and $p(1) = x_1$ and $p(2) = x_2$ and $p(3) = x_3$ and $p(4) = x_4$ and $p(5) = x_5$ and $p(6) = x_6$ and $p(7) = x_7$ and $p(8) = x_8$ and $p(9) = x_9$.

(14) Let I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9 be instructions of **SCM**, and let i_2, i_3, i_4, i_5 be integers, and let i_1 be a natural number, and let s be a state with instruction counter on i_1, with $\langle I_1 \rangle \frown \langle I_2 \rangle \frown \langle I_3 \rangle \frown \langle I_4 \rangle \frown \langle I_5 \rangle \frown \langle I_6 \rangle \frown \langle I_7 \rangle \frown \langle I_8 \rangle \frown \langle I_9 \rangle$ located from 0, and $\langle i_2 \rangle \frown \langle i_3 \rangle \frown \langle i_4 \rangle \frown \langle i_5 \rangle$ from 0. Then

(i) $\mathbf{IC}_s = \mathbf{i}_{(i_1)}$,

(ii) $s(\mathbf{i}_0) = I_1$,

(iii) $s(\mathbf{i}_1) = I_2$,

(iv) $s(\mathbf{i}_2) = I_3$,

(v) $s(\mathbf{i}_3) = I_4$,

(vi) $s(\mathbf{i}_4) = I_5$,

(vii) $s(\mathbf{i}_5) = I_6$,

(viii) $s(\mathbf{i}_6) = I_7$,
(ix) $s(\mathbf{i}_7) = I_8$,
(x) $s(\mathbf{i}_8) = I_9$,
(xi) $s(\mathbf{d}_0) = i_2$,
(xii) $s(\mathbf{d}_1) = i_3$,
(xiii) $s(\mathbf{d}_2) = i_4$, and
(xiv) $s(\mathbf{d}_3) = i_5$.

(15) Let I_1, I_2 be instructions of **SCM**, and let i_2, i_3 be integers, and let i_1 be a natural number, and let s be a state with instruction counter on i_1, with $\langle I_1 \rangle \frown \langle I_2 \rangle$ located from 0, and $\langle i_2 \rangle \frown \langle i_3 \rangle$ from 0. Then $\mathbf{IC}_s = \mathbf{i}_{(i_1)}$ and $s(\mathbf{i}_0) = I_1$ and $s(\mathbf{i}_1) = I_2$ and $s(\mathbf{d}_0) = i_2$ and $s(\mathbf{d}_1) = i_3$.

Let a, b be data-locations. Then $a{:=}b$ is an instruction of **SCM**. Then AddTo(a, b) is an instruction of **SCM**. Then SubFrom(a, b) is an instruction of **SCM**. Then MultBy(a, b) is an instruction of **SCM**. Then Divide(a, b) is an instruction of **SCM**.

Let l_1 be an instruction-location of **SCM**. Then goto l_1 is an instruction of **SCM**. Let a be a data-location. Then **if** $a = 0$ **goto** l_1 is an instruction of **SCM**. Then **if** $a > 0$ **goto** l_1 is an instruction of **SCM**.

Let s be a state of **SCM**. Let us assume that s is halting. The complexity of s is a natural number and is defined by the conditions (Def.2).

(Def.2) (i) CurInstr((Computation(s))(the complexity of s)) = $\mathbf{halt}_{\mathbf{SCM}}$, and
(ii) for every natural number k such that CurInstr((Computation(s))(k)) = $\mathbf{halt}_{\mathbf{SCM}}$ holds the complexity of $s \leq k$.

We now state a number of propositions:

(16) Let s be a state of **SCM** and let k be a natural number. Then $s(\mathbf{IC}_{(\mathrm{Computation}(s))(k)}) \neq \mathbf{halt}_{\mathbf{SCM}}$ and $s(\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)}) = \mathbf{halt}_{\mathbf{SCM}}$ if and only if the complexity of $s = k + 1$ and s is halting.

(17) Let s be a state of **SCM** and let k be a natural number. If $\mathbf{IC}_{(\mathrm{Computation}(s))(k)} \neq \mathbf{IC}_{(\mathrm{Computation}(s))(k+1)}$ and $s(\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)}) = \mathbf{halt}_{\mathbf{SCM}}$, then the complexity of $s = k + 1$.

(18) Let k, n be natural numbers, and let s be a state of **SCM**, and let a, b be data-locations. Suppose $\mathbf{IC}_{(\mathrm{Computation}(s))(k)} = \mathbf{i}_n$ and $s(\mathbf{i}_n) = a{:=}b$. Then $\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)} = \mathbf{i}_{n+1}$ and (Computation(s))($k + 1$)(a) = (Computation(s))(k)(b) and for every data-location d such that $d \neq a$ holds (Computation(s))($k + 1$)(d) = (Computation(s))(k)(d).

(19) Let k, n be natural numbers, and let s be a state of **SCM**, and let a, b be data-locations. Suppose $\mathbf{IC}_{(\mathrm{Computation}(s))(k)} = \mathbf{i}_n$ and $s(\mathbf{i}_n) = \mathrm{AddTo}(a, b)$. Then $\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)} = \mathbf{i}_{n+1}$ and (Computation(s))($k + 1$)(a) = (Computation(s))(k)(a) + (Computation(s))(k)(b) and for every data-location d such that $d \neq a$ holds (Computation(s))($k + 1$)(d) = (Computation(s))(k)(d).

(20) Let k, n be natural numbers, and let s be a state of **SCM**, and let a, b be data-locations. Suppose $\mathbf{IC}_{(\mathrm{Computation}(s))(k)} =$

$\mathbf{i}_n$ and $s(\mathbf{i}_n) = \mathrm{SubFrom}(a,b)$. Then $\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)} = \mathbf{i}_{n+1}$ and $(\mathrm{Computation}(s))(k+1)(a) = (\mathrm{Computation}(s))(k)(a) - (\mathrm{Computation}(s))(k)(b)$ and for every data-location d such that $d \neq a$ holds $(\mathrm{Computation}(s))(k+1)(d) = (\mathrm{Computation}(s))(k)(d)$.

(21) Let k, n be natural numbers, and let s be a state of **SCM**, and let a, b be data-locations. Suppose $\mathbf{IC}_{(\mathrm{Computation}(s))(k)} = \mathbf{i}_n$ and $s(\mathbf{i}_n) = \mathrm{MultBy}(a,b)$. Then $\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)} = \mathbf{i}_{n+1}$ and $(\mathrm{Computation}(s))(k+1)(a) = (\mathrm{Computation}(s))(k)(a) \cdot (\mathrm{Computation}(s))(k)(b)$ and for every data-location d such that $d \neq a$ holds $(\mathrm{Computation}(s))(k+1)(d) = (\mathrm{Computation}(s))(k)(d)$.

(22) Let k, n be natural numbers, and let s be a state of **SCM**, and let a, b be data-locations. Suppose $\mathbf{IC}_{(\mathrm{Computation}(s))(k)} = \mathbf{i}_n$ and $s(\mathbf{i}_n) = \mathrm{Divide}(a,b)$ and $a \neq b$. Then

(i) $\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)} = \mathbf{i}_{n+1}$,

(ii) $(\mathrm{Computation}(s))(k+1)(a) = (\mathrm{Computation}(s))(k)(a) \div (\mathrm{Computation}(s))(k)(b)$,

(iii) $(\mathrm{Computation}(s))(k+1)(b) = (\mathrm{Computation}(s))(k)(a) \bmod (\mathrm{Computation}(s))(k)(b)$, and

(iv) for every data-location d such that $d \neq a$ and $d \neq b$ holds $(\mathrm{Computation}(s))(k+1)(d) = (\mathrm{Computation}(s))(k)(d)$.

(23) Let k, n be natural numbers, and let s be a state of **SCM**, and let i_1 be an instruction-location of **SCM**. Suppose $\mathbf{IC}_{(\mathrm{Computation}(s))(k)} = \mathbf{i}_n$ and $s(\mathbf{i}_n) = \mathrm{goto}\ i_1$. Then $\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)} = i_1$ and for every data-location d holds $(\mathrm{Computation}(s))(k+1)(d) = (\mathrm{Computation}(s))(k)(d)$.

(24) Let k, n be natural numbers, and let s be a state of **SCM**, and let a be a data-location, and let i_1 be an instruction-location of **SCM**. Suppose $\mathbf{IC}_{(\mathrm{Computation}(s))(k)} = \mathbf{i}_n$ and $s(\mathbf{i}_n) = \mathbf{if}\ a = 0\ \mathbf{goto}\ i_1$. Then

(i) if $(\mathrm{Computation}(s))(k)(a) = 0$, then $\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)} = i_1$,

(ii) if $(\mathrm{Computation}(s))(k)(a) \neq 0$, then $\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)} = \mathbf{i}_{n+1}$, and

(iii) for every data-location d holds $(\mathrm{Computation}(s))(k+1)(d) = (\mathrm{Computation}(s))(k)(d)$.

(25) Let k, n be natural numbers, and let s be a state of **SCM**, and let a be a data-location, and let i_1 be an instruction-location of **SCM**. Suppose $\mathbf{IC}_{(\mathrm{Computation}(s))(k)} = \mathbf{i}_n$ and $s(\mathbf{i}_n) = \mathbf{if}\ a > 0\ \mathbf{goto}\ i_1$. Then

(i) if $(\mathrm{Computation}(s))(k)(a) > 0$, then $\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)} = i_1$,

(ii) if $(\mathrm{Computation}(s))(k)(a) \leq 0$, then $\mathbf{IC}_{(\mathrm{Computation}(s))(k+1)} = \mathbf{i}_{n+1}$, and

(iii) for every data-location d holds $(\mathrm{Computation}(s))(k+1)(d) = (\mathrm{Computation}(s))(k)(d)$.

(26) (i) $(\mathbf{halt}_{\mathbf{SCM}})_1 = 0$,

(ii) for all data-locations a, b holds $(a{:=}b)_1 = 1$,

(iii) for all data-locations a, b holds $(\mathrm{AddTo}(a,b))_1 = 2$,

(iv) for all data-locations a, b holds $(\mathrm{SubFrom}(a,b))_1 = 3$,

(v) for all data-locations a, b holds $(\mathrm{MultBy}(a,b))_1 = 4$,

(vi) for all data-locations a, b holds $(\mathrm{Divide}(a,b))_1 = 5$,

(vii) for every instruction-location i of **SCM** holds $(\text{goto } i)_1 = 6$,

(viii) for every data-location a and for every instruction-location i of **SCM** holds $(\textbf{if } a = 0 \textbf{ goto } i)_1 = 7$, and

(ix) for every data-location a and for every instruction-location i of **SCM** holds $(\textbf{if } a > 0 \textbf{ goto } i)_1 = 8$.

(27) For all states s_1, s_2 of **SCM** and for every natural number k such that $s_2 = (\text{Computation}(s_1))(k)$ and s_2 is halting holds s_1 is halting.

(28) Let s_1, s_2 be states of **SCM** and let k, c be natural numbers. Suppose $s_2 = (\text{Computation}(s_1))(k)$ and the complexity of $s_2 = c$ and s_2 is halting and $0 < c$. Then the complexity of $s_1 = k + c$.

(29) For all states s_1, s_2 of **SCM** and for every natural number k such that $s_2 = (\text{Computation}(s_1))(k)$ and s_2 is halting holds $\text{Result}(s_2) = \text{Result}(s_1)$.

(30) Let I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9 be instructions of **SCM**, and let i_2, i_3, i_4, i_5 be integers, and let i_1 be a natural number, and let s be a state of **SCM**. Suppose that

(i) $\mathbf{IC}_s = \mathbf{i}_{(i_1)}$,

(ii) $s(\mathbf{i}_0) = I_1$,

(iii) $s(\mathbf{i}_1) = I_2$,

(iv) $s(\mathbf{i}_2) = I_3$,

(v) $s(\mathbf{i}_3) = I_4$,

(vi) $s(\mathbf{i}_4) = I_5$,

(vii) $s(\mathbf{i}_5) = I_6$,

(viii) $s(\mathbf{i}_6) = I_7$,

(ix) $s(\mathbf{i}_7) = I_8$,

(x) $s(\mathbf{i}_8) = I_9$,

(xi) $s(\mathbf{d}_0) = i_2$,

(xii) $s(\mathbf{d}_1) = i_3$,

(xiii) $s(\mathbf{d}_2) = i_4$, and

(xiv) $s(\mathbf{d}_3) = i_5$.

Then s is a state with instruction counter on i_1, with $\langle I_1\rangle \frown \langle I_2\rangle \frown \langle I_3\rangle \frown \langle I_4\rangle \frown \langle I_5\rangle \frown \langle I_6\rangle \frown \langle I_7\rangle \frown \langle I_8\rangle \frown \langle I_9\rangle$ located from 0, and $\langle i_2\rangle \frown \langle i_3\rangle \frown \langle i_4\rangle \frown \langle i_5\rangle$ from 0.

(31) Let s be a state with instruction counter on 0, with $\langle \mathbf{halt}_{\mathbf{SCM}}\rangle$ located from 0, and $\varepsilon_{\mathbb{Z}}$ from 0. Then s is halting and the complexity of $s = 0$ and $\text{Result}(s) = s$.

(32) Let i_2, i_3 be integers and let s be a state with instruction counter on 0, with $\langle \mathbf{d}_0{:=}\mathbf{d}_1\rangle \frown \langle \mathbf{halt}_{\mathbf{SCM}}\rangle$ located from 0, and $\langle i_2\rangle \frown \langle i_3\rangle$ from 0. Then

(i) s is halting,

(ii) the complexity of $s = 1$,

(iii) $(\text{Result}(s))(\mathbf{d}_0) = i_3$, and

(iv) for every data-location d such that $d \neq \mathbf{d}_0$ holds $(\text{Result}(s))(d) = s(d)$.

(33) Let i_2, i_3 be integers and let s be a state with instruction counter on 0, with $\langle \mathrm{AddTo}(\mathbf{d}_0, \mathbf{d}_1) \rangle \frown \langle \mathbf{halt}_{\mathbf{SCM}} \rangle$ located from 0, and $\langle i_2 \rangle \frown \langle i_3 \rangle$ from 0. Then
- (i) s is halting,
- (ii) the complexity of $s = 1$,
- (iii) $(\mathrm{Result}(s))(\mathbf{d}_0) = i_2 + i_3$, and
- (iv) for every data-location d such that $d \neq \mathbf{d}_0$ holds $(\mathrm{Result}(s))(d) = s(d)$.

(34) Let i_2, i_3 be integers and let s be a state with instruction counter on 0, with $\langle \mathrm{SubFrom}(\mathbf{d}_0, \mathbf{d}_1) \rangle \frown \langle \mathbf{halt}_{\mathbf{SCM}} \rangle$ located from 0, and $\langle i_2 \rangle \frown \langle i_3 \rangle$ from 0. Then
- (i) s is halting,
- (ii) the complexity of $s = 1$,
- (iii) $(\mathrm{Result}(s))(\mathbf{d}_0) = i_2 - i_3$, and
- (iv) for every data-location d such that $d \neq \mathbf{d}_0$ holds $(\mathrm{Result}(s))(d) = s(d)$.

(35) Let i_2, i_3 be integers and let s be a state with instruction counter on 0, with $\langle \mathrm{MultBy}(\mathbf{d}_0, \mathbf{d}_1) \rangle \frown \langle \mathbf{halt}_{\mathbf{SCM}} \rangle$ located from 0, and $\langle i_2 \rangle \frown \langle i_3 \rangle$ from 0. Then
- (i) s is halting,
- (ii) the complexity of $s = 1$,
- (iii) $(\mathrm{Result}(s))(\mathbf{d}_0) = i_2 \cdot i_3$, and
- (iv) for every data-location d such that $d \neq \mathbf{d}_0$ holds $(\mathrm{Result}(s))(d) = s(d)$.

(36) Let i_2, i_3 be integers and let s be a state with instruction counter on 0, with $\langle \mathrm{Divide}(\mathbf{d}_0, \mathbf{d}_1) \rangle \frown \langle \mathbf{halt}_{\mathbf{SCM}} \rangle$ located from 0, and $\langle i_2 \rangle \frown \langle i_3 \rangle$ from 0. Then
- (i) s is halting,
- (ii) the complexity of $s = 1$,
- (iii) $(\mathrm{Result}(s))(\mathbf{d}_0) = i_2 \div i_3$,
- (iv) $(\mathrm{Result}(s))(\mathbf{d}_1) = i_2 \bmod i_3$, and
- (v) for every data-location d such that $d \neq \mathbf{d}_0$ and $d \neq \mathbf{d}_1$ holds $(\mathrm{Result}(s))(d) = s(d)$.

(37) Let i_2, i_3 be integers and let s be a state with instruction counter on 0, with $\langle \mathbf{goto}\ (\mathbf{i}_1) \rangle \frown \langle \mathbf{halt}_{\mathbf{SCM}} \rangle$ located from 0, and $\langle i_2 \rangle \frown \langle i_3 \rangle$ from 0. Then s is halting and the complexity of $s = 1$ and for every data-location d holds $(\mathrm{Result}(s))(d) = s(d)$.

(38) Let i_2, i_3 be integers and let s be a state with instruction counter on 0, with $\langle \mathbf{if}\ \mathbf{d}_0 = 0\ \mathbf{goto}\ \mathbf{i}_1 \rangle \frown \langle \mathbf{halt}_{\mathbf{SCM}} \rangle$ located from 0, and $\langle i_2 \rangle \frown \langle i_3 \rangle$ from 0. Then s is halting and the complexity of $s = 1$ and for every data-location d holds $(\mathrm{Result}(s))(d) = s(d)$.

(39) Let i_2, i_3 be integers and let s be a state with instruction counter on 0, with $\langle \mathbf{if}\ \mathbf{d}_0 > 0\ \mathbf{goto}\ \mathbf{i}_1 \rangle \frown \langle \mathbf{halt}_{\mathbf{SCM}} \rangle$ located from 0, and $\langle i_2 \rangle \frown \langle i_3 \rangle$ from 0. Then s is halting and the complexity of $s = 1$ and for every data-location d holds $(\mathrm{Result}(s))(d) = s(d)$.

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Two Programs for SCM. Part I - Preliminaries [1]

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Summary. In two articles (this one and [3]) we discuss correctness of two short programs for the **SCM** machine: one computes Fibonacci numbers and the other computes the *fusc* function of Dijkstra [7]. The limitations of current Mizar implementation rendered it impossible to present the correctness proofs for the programs in one article. This part is purely technical and contains a number of very specific lemmas about integer division, floor, exponentiation and logarithms. The formal definitions of the Fibonacci sequence and the *fusc* function may be of general interest.

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The terminology and notation used in this paper are introduced in the following papers: [12], [1], [14], [9], [13], [11], [10], [8], [5], [6], [2], [4], and [15].

Let X_1, X_2 be non empty set, let Y_1 be a non empty subset of X_1, and let Y_2 be a non empty subset of X_2. Then $[: Y_1, Y_2 :]$ is a non empty subset of $[: X_1, X_2 :]$.

Let X_1, X_2 be non empty set, let Y_1 be a non empty subset of X_1, let Y_2 be a non empty subset of X_2, and let x be an element of $[: Y_1, Y_2 :]$. Then x_1 is an element of Y_1. Then x_2 is an element of Y_2.

In the sequel n will denote a natural number.

Let us consider n. The functor $\mathrm{Fib}(n)$ yielding a natural number is defined by the condition (Def.1).

(Def.1) There exists a function f_1 from $\mathbb{N}$ into $[: \mathbb{N}, \mathbb{N} :]$ such that

(i) $\mathrm{Fib}(n) = f_1(n)_1$,

(ii) $f_1(0) = \langle 0, 1 \rangle$, and

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(iii) for every natural number n and for every element x of $[:\mathbb{N}, \mathbb{N}:]$ such that $x = f_1(n)$ holds $f_1(n+1) = \langle x_2, x_1 + x_2 \rangle$.

We now state a number of propositions:

(1) $\mathrm{Fib}(0) = 0$ and $\mathrm{Fib}(1) = 1$ and for every natural number n holds $\mathrm{Fib}(n+1+1) = \mathrm{Fib}(n) + \mathrm{Fib}(n+1)$.

(2) For every integer i holds $i \div +1 = i$.

(3) For all integers i, j such that $j > 0$ and $i \div j = 0$ holds $i < j$.

(4) For all integers i, j such that $0 \leq i$ and $i < j$ holds $i \div j = 0$.

(5) For all integers i, j, k such that $j > 0$ and $k > 0$ holds $i \div j \div k = i \div j \cdot k$.

(6) For every integer i holds $i \bmod +2 = 0$ or $i \bmod +2 = 1$.

(7) For every integer i such that i is a natural number holds $i \div +2$ is a natural number.

(8) For every natural number k such that $k > 0$ and for every natural number n holds $k^n > 0$.

(9)[2] For every natural number n holds $2^n = 2^n$.

(10) For all real numbers a, b, c such that $a \leq b$ and $c > 1$ holds $c^a \leq c^b$.

Let a, n be natural numbers. Then a^n is a natural number.

Next we state several propositions:

(11) For all real numbers r, s such that $r \geq s$ holds $\lfloor r \rfloor \geq \lfloor s \rfloor$.

(12) For all real numbers a, b, c such that $a > 1$ and $b > 0$ and $c \geq b$ holds $\log_a c \geq \log_a b$.

(13) For every natural number n such that $n > 0$ holds $\lfloor \log_2(2 \cdot n) \rfloor + 1 \neq \lfloor \log_2(2 \cdot n + 1) \rfloor$.

(14) For every natural number n such that $n > 0$ holds $\lfloor \log_2(2 \cdot n) \rfloor + 1 \geq \lfloor \log_2(2 \cdot n + 1) \rfloor$.

(15) For every natural number n such that $n > 0$ holds $\lfloor \log_2(2 \cdot n) \rfloor = \lfloor \log_2(2 \cdot n + 1) \rfloor$.

(16) For every natural number n such that $n > 0$ holds $\lfloor \log_2 n \rfloor + 1 = \lfloor \log_2(2 \cdot n + 1) \rfloor$.

Let f be a function from $\mathbb{N}$ into $\mathbb{N}^*$ and let n be a natural number. Then $f(n)$ is a finite sequence of elements of $\mathbb{N}$.

Let n be a natural number. The functor $\mathrm{Fusc}(n)$ yields a natural number and is defined by:

(Def.2) (i) $\mathrm{Fusc}(n) = 0$ if $n = 0$,

(ii) there exists a natural number l and there exists a function f_2 from $\mathbb{N}$ into $\mathbb{N}^*$ such that $l+1 = n$ and $\mathrm{Fusc}(n) = \pi_n f_2(l)$ and $f_2(0) = \langle 1 \rangle$ and for every natural number n holds for every natural number k such that $n+2 = 2 \cdot k$ holds $f_2(n+1) = f_2(n) \frown \langle \pi_k f_2(n) \rangle$ and for every natural number k such that $n+2 = 2 \cdot k + 1$ holds $f_2(n+1) = f_2(n) \frown \langle \pi_k f_2(n) + \pi_{k+1} f_2(n) \rangle$, otherwise.

[2]Both power functions in this theorem are different. The first is defined in [10] and the second in [8].

The following propositions are true:

(17) $\mathrm{Fusc}(0) = 0$ and $\mathrm{Fusc}(1) = 1$ and for every natural number n holds $\mathrm{Fusc}(2 \cdot n) = \mathrm{Fusc}(n)$ and $\mathrm{Fusc}(2 \cdot n + 1) = \mathrm{Fusc}(n) + \mathrm{Fusc}(n + 1)$.

(18) For all natural numbers n_1, n_1' such that $n_1 \neq 0$ and $n_1 = 2 \cdot n_1'$ holds $n_1' < n_1$.

(19) For all natural numbers n_1, n_1' such that $n_1 = 2 \cdot n_1' + 1$ holds $n_1' < n_1$.

(20) For all natural numbers A, B holds $B = A \cdot \mathrm{Fusc}(0) + B \cdot \mathrm{Fusc}(0 + 1)$.

(21) For all natural numbers n_1, n_1', A, B, N such that $n_1 = 2 \cdot n_1' + 1$ and $\mathrm{Fusc}(N) = A \cdot \mathrm{Fusc}(n_1) + B \cdot \mathrm{Fusc}(n_1 + 1)$ holds $\mathrm{Fusc}(N) = A \cdot \mathrm{Fusc}(n_1') + (B + A) \cdot \mathrm{Fusc}(n_1' + 1)$.

(22) For all natural numbers n_1, n_1', A, B, N such that $n_1 = 2 \cdot n_1'$ and $\mathrm{Fusc}(N) = A \cdot \mathrm{Fusc}(n_1) + B \cdot \mathrm{Fusc}(n_1 + 1)$ holds $\mathrm{Fusc}(N) = (A + B) \cdot \mathrm{Fusc}(n_1') + B \cdot \mathrm{Fusc}(n_1' + 1)$.

(23) $6 + 1 = 6 \cdot (\lfloor \log_2 1 \rfloor + 1) + 1$.

(24) For every natural number n_1' such that $n_1' > 0$ holds $\lfloor \log_2 n_1' \rfloor$ is a natural number and $6 \cdot (\lfloor \log_2 n_1' \rfloor + 1) + 1 > 0$.

(25) For all natural numbers n_1, n_1' such that $n_1 = 2 \cdot n_1' + 1$ and $n_1' > 0$ holds $6 + (6 \cdot (\lfloor \log_2 n_1' \rfloor + 1) + 1) = 6 \cdot (\lfloor \log_2 n_1 \rfloor + 1) + 1$.

(26) For all natural numbers n_1, n_1' such that $n_1 = 2 \cdot n_1'$ and $n_1' > 0$ holds $6 + (6 \cdot (\lfloor \log_2 n_1' \rfloor + 1) + 1) = 6 \cdot (\lfloor \log_2 n_1 \rfloor + 1) + 1$.

(27) For every natural number N such that $N \neq 0$ holds $6 \cdot N - 4 > 0$.

(28) For every natural number N holds $6 + (6 \cdot N - 4) = 6 \cdot (N + 1) - 4$.

(29) For all natural numbers m, k, N such that $m = (k + 1 + N) - 1$ holds $m = (k + (N + 1)) - 1$.

(30) For every natural number N holds $2 + (6 \cdot N - 4) = 6 \cdot N - 2$.

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Two Programs for SCM. Part II - Programs [1]

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Summary. We prove the correctness of two short programs for the **SCM** machine: one computes Fibonacci numbers and the other computes the *fusc* function of Dijkstra [11]. The formal definitions of these functions can be found in [5]. We prove the total correctness of the programs in two ways: by conducting inductions on computations and inductions on input data. In addition we characterize the concrete complexity of the programs as defined in [4].

MML Identifier: FIB_FUSC.

The papers [17], [1], [20], [13], [18], [10], [16], [12], [7], [8], [2], [3], [6], [21], [9], [14], [15], [4], [19], and [5] provide the terminology and notation for this paper.

The program computing Fib is a finite sequence of elements of the instructions of **SCM** and is defined as follows:

(Def.1) The program computing Fib = $\langle \textbf{if } \mathbf{d}_1 > 0 \textbf{ goto } \mathbf{i}_2 \rangle \frown \langle \mathbf{halt}_{\mathbf{SCM}} \rangle \frown \langle \mathbf{d}_3{:=}\mathbf{d}_0 \rangle \frown \langle \mathrm{SubFrom}(\mathbf{d}_1, \mathbf{d}_0) \rangle \frown \langle \textbf{if } \mathbf{d}_1 = 0 \textbf{ goto } \mathbf{i}_1 \rangle \frown \langle \mathbf{d}_4{:=}\mathbf{d}_2 \rangle \frown \langle \mathbf{d}_2{:=}\mathbf{d}_3 \rangle \frown \langle \mathrm{AddTo}(\mathbf{d}_3, \mathbf{d}_4) \rangle \frown \langle \mathrm{goto}\ (\mathbf{i}_3) \rangle$.

The following proposition is true

(1) Let N be a natural number and let s be a state with instruction counter on 0, with the program computing Fib located from 0, and $\langle +1 \rangle \frown \langle +N \rangle \frown \langle +0 \rangle \frown \langle +0 \rangle$ from 0. Then
- (i) s is halting,
- (ii) if $N = 0$, then the complexity of $s = 1$,
- (iii) if $N > 0$, then the complexity of $s = 6 \cdot N - 2$, and
- (iv) $(\mathrm{Result}(s))(\mathbf{d}_3) = \mathrm{Fib}(N)$.

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Let i be an integer. The functor $\mathrm{Fusc}(i)$ yields a natural number and is defined as follows:

(Def.2) There exists a natural number n such that $i = n$ and $\mathrm{Fusc}(i) = \mathrm{Fusc}(n)$ or i is not a natural number and $\mathrm{Fusc}(i) = 0$.

Let a, n be natural numbers. Then a^n is an integer.

The program computing Fusc is a finite sequence of elements of the instructions of **SCM** and is defined by:

(Def.3) The program computing Fusc $= \langle \textbf{if } \mathbf{d}_1 = 0 \textbf{ goto } \mathbf{i}_8 \rangle \frown \langle \mathbf{d}_4{:=}\mathbf{d}_0 \rangle \frown \langle \mathrm{Divide}(\mathbf{d}_1, \mathbf{d}_4) \rangle \frown \langle \textbf{if } \mathbf{d}_4 = 0 \textbf{ goto } \mathbf{i}_6 \rangle \frown \langle \mathrm{AddTo}(\mathbf{d}_3, \mathbf{d}_2) \rangle \frown \langle \mathrm{goto}\ (\mathbf{i}_0) \rangle \frown \langle \mathrm{AddTo}(\mathbf{d}_2, \mathbf{d}_3) \rangle \frown \langle \mathrm{goto}\ (\mathbf{i}_0) \rangle \frown \langle \mathbf{halt}_{\mathbf{SCM}} \rangle$.

We now state several propositions:

(2) Let N be a natural number. Suppose $N > 0$. Let s be a state with instruction counter on 0, with the program computing Fusc located from 0, and $\langle +2 \rangle \frown \langle +N \rangle \frown \langle +1 \rangle \frown \langle +0 \rangle$ from 0. Then s is halting and $(\mathrm{Result}(s))(\mathbf{d}_3) = \mathrm{Fusc}(N)$ and the complexity of $s = 6 \cdot (\lfloor \log_2 N \rfloor + 1) + 1$.

(3) Let N be a natural number, and let k, F_1, F_2 be natural numbers, and let s be a state with instruction counter on 3, with the program computing Fib located from 0, and $\langle +1 \rangle \frown \langle +N \rangle \frown \langle +F_1 \rangle \frown \langle +F_2 \rangle$ from 0. Suppose $N > 0$ and $F_1 = \mathrm{Fib}(k)$ and $F_2 = \mathrm{Fib}(k+1)$. Then

(i) s is halting,

(ii) the complexity of $s = 6 \cdot N - 4$, and

(iii) there exists a natural number m such that $m = (k + N) - 1$ and $(\mathrm{Result}(s))(\mathbf{d}_2) = \mathrm{Fib}(m)$ and $(\mathrm{Result}(s))(\mathbf{d}_3) = \mathrm{Fib}(m+1)$.

(4) Let N be a natural number and let s be a state with instruction counter on 0, with the program computing Fib located from 0, and $\langle +1 \rangle \frown \langle +N \rangle \frown \langle +0 \rangle \frown \langle +0 \rangle$ from 0. Then

(i) s is halting,

(ii) if $N = 0$, then the complexity of $s = 1$,

(iii) if $N > 0$, then the complexity of $s = 6 \cdot N - 2$, and

(iv) $(\mathrm{Result}(s))(\mathbf{d}_3) = \mathrm{Fib}(N)$.

(5) Let n be a natural number, and let N, A, B be natural numbers, and let s be a state with instruction counter on 0, with the program computing Fusc located from 0, and $\langle +2 \rangle \frown \langle +n \rangle \frown \langle +A \rangle \frown \langle +B \rangle$ from 0. Suppose $N > 0$ and $\mathrm{Fusc}(N) = A \cdot \mathrm{Fusc}(n) + B \cdot \mathrm{Fusc}(n+1)$. Then

(i) s is halting,

(ii) $(\mathrm{Result}(s))(\mathbf{d}_3) = \mathrm{Fusc}(N)$,

(iii) if $n = 0$, then the complexity of $s = 1$, and

(iv) if $n > 0$, then the complexity of $s = 6 \cdot (\lfloor \log_2 n \rfloor + 1) + 1$.

(6) Let N be a natural number. Suppose $N > 0$. Let s be a state with instruction counter on 0, with the program computing Fusc located from 0, and $\langle +2 \rangle \frown \langle +N \rangle \frown \langle +1 \rangle \frown \langle +0 \rangle$ from 0. Then

(i) s is halting,

(ii) $(\mathrm{Result}(s))(\mathbf{d}_3) = \mathrm{Fusc}(N)$,

(iii) if $N = 0$, then the complexity of $s = 1$, and
(iv) if $N > 0$, then the complexity of $s = 6 \cdot (\lfloor \log_2 N \rfloor + 1) + 1$.

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Joining of Decorated Trees

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Summary. This is the continuation of the sequence of articles on trees (see [3,4,5]). The main goal is to introduce joining operations on decorated trees corresponding with operations introduced in [5]. We will also introduce the operation of substitution. In the last section we dealt with trees decorated by Cartesian product, i.e. we showed some lemmas on joining operations applied to such trees.

MML Identifier: TREES_4.

The notation and terminology used here are introduced in the following papers: [15], [2], [9], [16], [11], [14], [13], [12], [10], [7], [6], [8], [3], [4], [1], and [5].

1. Joining of Decorated Tree

Let T be a decorated tree. A node of T is an element of $\operatorname{dom} T$.

We adopt the following convention: x, y, z are arbitrary, i, j, n denote natural numbers, and p, q denote finite sequences.

Let T_1, T_2 be decorated trees. Let us observe that $T_1 = T_2$ if and only if:

(Def.1) $\operatorname{dom} T_1 = \operatorname{dom} T_2$ and for every node p of T_1 holds $T_1(p) = T_2(p)$.

One can prove the following two propositions:

(1) For all natural numbers i, j such that the elementary tree of $i \subseteq$ the elementary tree of j holds $i \leq j$.

(2) For all natural numbers i, j such that the elementary tree of $i =$ the elementary tree of j holds $i = j$.

Let us consider x. The root tree of x is a decorated tree and is defined as follows:

(Def.2) The root tree of $x =$ (the elementary tree of $0) \longmapsto x$.

Let D be a non empty set and let d be an element of D. Then the root tree of d is an element of $\operatorname{FinTrees}(D)$.

We now state four propositions:

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(3) dom (the root tree of x) = the elementary tree of 0 and (the root tree of x)(ε) = x.

(4) If the root tree of x = the root tree of y, then $x = y$.

(5) For every decorated tree T such that dom T = the elementary tree of 0 holds T = the root tree of $T(\varepsilon)$.

(6) The root tree of $x = \{\langle\varepsilon, x\rangle\}$.

Let us consider x and let p be a finite sequence. The flat tree of x and p is a decorated tree and is defined by the conditions (Def.3).

(Def.3) (i) dom (the flat tree of x and p) = the elementary tree of len p,

(ii) (the flat tree of x and p)(ε) = x, and

(iii) for every n such that $n <$ len p holds (the flat tree of x and p)($\langle n\rangle$) = $p(n+1)$.

The following propositions are true:

(7) If the flat tree of x and p = the flat tree of y and q, then $x = y$ and $p = q$.

(8) If $j < i$, then (the elementary tree of i) $\restriction \langle j\rangle$ = the elementary tree of 0.

(9) If $i <$ len p, then (the flat tree of x and p) $\restriction \langle i\rangle$ = the root tree of $p(i+1)$.

Let us consider x, p. Let us assume that p is decorated tree yielding. The functor x-tree(p) yields a decorated tree and is defined by the conditions (Def.4).

(Def.4) (i) There exists a decorated tree yielding finite sequence q such that $p = q$ and dom(x-tree(p)) = $\widehat{\operatorname{dom}_{\kappa} q(\kappa)}$,

(ii) (x-tree(p))(ε) = x, and

(iii) for every n such that $n <$ len p holds (x-tree(p)) $\restriction \langle n\rangle = p(n+1)$.

Let us consider x and let T be a decorated tree. The functor x-tree(T) yielding a decorated tree is defined by:

(Def.5) x-tree(T) = x-tree($\langle T\rangle$).

Let us consider x and let T_1, T_2 be decorated trees. The functor x-tree(T_1, T_2) yields a decorated tree and is defined as follows:

(Def.6) x-tree(T_1, T_2) = x-tree($\langle T_1, T_2\rangle$).

We now state a number of propositions:

(10) For every decorated tree yielding finite sequence p holds dom(x-tree(p)) = $\widehat{\operatorname{dom}_{\kappa} p(\kappa)}$.

(11) Let p be a decorated tree yielding finite sequence. Then $y \in$ dom(x-tree(p)) if and only if one of the following conditions is satisfied:

(i) $y = \varepsilon$, or

(ii) there exists a natural number i and there exists a decorated tree T and there exists a node q of T such that $i <$ len p and $T = p(i+1)$ and $y = \langle i\rangle \frown q$.

(12) Let p be a decorated tree yielding finite sequence, and let i be a natural number, and let T be a decorated tree, and let q be a node of T. If

$i < \operatorname{len} p$ and $T = p(i+1)$, then $(x\text{-tree}(p))(\langle i \rangle ^\frown q) = T(q)$.

(13) For every decorated tree T holds $\operatorname{dom}(x\text{-tree}(T)) = \widehat{\operatorname{dom} T}$.

(14) For all decorated trees T_1, T_2 holds $\operatorname{dom}(x\text{-tree}(T_1, T_2)) = \widehat{\operatorname{dom} T_1, \operatorname{dom} T_2}$.

(15) For all decorated tree yielding finite sequence p, q such that $x\text{-tree}(p) = y\text{-tree}(q)$ holds $x = y$ and $p = q$.

(16) If the root tree of $x =$ the flat tree of y and p, then $x = y$ and $p = \varepsilon$.

(17) If the root tree of $x = y\text{-tree}(p)$ and p is decorated tree yielding, then $x = y$ and $p = \varepsilon$.

(18) Suppose the flat tree of x and $p = y\text{-tree}(q)$ and q is decorated tree yielding. Then $x = y$ and $\operatorname{len} p = \operatorname{len} q$ and for every i such that $i \in \operatorname{dom} p$ holds $q(i) =$ the root tree of $p(i)$.

(19) Let p be a decorated tree yielding finite sequence, and let n be a natural number, and let q be a finite sequence. If $\langle n \rangle ^\frown q \in \operatorname{dom}(x\text{-tree}(p))$, then $(x\text{-tree}(p))(\langle n \rangle ^\frown q) = p(n+1)(q)$.

(20) The flat tree of x and $\varepsilon =$ the root tree of x and $x\text{-tree}(\varepsilon) =$ the root tree of x.

(21) The flat tree of x and $\langle y \rangle =$ ((the elementary tree of 1) $\longmapsto x)(\langle 0 \rangle/$(the root tree of y)).

(22) For every decorated tree T holds $x\text{-tree}(\langle T \rangle) =$ ((the elementary tree of 1) $\longmapsto x)(\langle 0 \rangle / T)$.

Let D be a non empty set, let d be an element of D, and let p be a finite sequence of elements of D. Then the flat tree of d and p is a tree decorated by D.

Let D be a non empty set, let F be a non empty set of trees decorated by D, let d be an element of D, and let p be a finite sequence of elements of F. Then $d\text{-tree}(p)$ is a tree decorated by D.

Let D be a non empty set, let d be an element of D, and let T be a tree decorated by D. Then $d\text{-tree}(T)$ is a tree decorated by D.

Let D be a non empty set, let d be an element of D, and let T_1, T_2 be trees decorated by D. Then $d\text{-tree}(T_1, T_2)$ is a tree decorated by D.

Let D be a non empty set and let p be a finite sequence of elements of $\operatorname{FinTrees}(D)$. Then $\operatorname{dom}_\kappa p(\kappa)$ is a finite sequence of elements of FinTrees.

Let D be a non empty set, let d be an element of D, and let p be a finite sequence of elements of $\operatorname{FinTrees}(D)$. Then $d\text{-tree}(p)$ is an element of $\operatorname{FinTrees}(D)$.

Let D be a non empty set and let x be a subset of D. We see that the finite sequence of elements of x is a finite sequence of elements of D.

Let D be a non empty constituted of decorated trees set and let X be a subset of D. Note that every finite sequence of elements of X is decorated tree yielding.

2. Expanding of Decorated Tree by Substitution

The scheme *ExpandTree* concerns a tree $\mathcal{A}$, a tree $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a tree T such that for every p holds $p \in T$ if and only if one of the following conditions is satisfied:

(i) $p \in \mathcal{A}$, or

(ii) there exists an element q of $\mathcal{A}$ and there exists an element r of $\mathcal{B}$ such that $\mathcal{P}[q]$ and $p = q \frown r$

for all values of the parameters.

Let T, T' be decorated trees and let x be arbitrary. The functor $T_{x \leftarrow T'}$ yielding a decorated tree is defined by the conditions (Def.7).

(Def.7) (i) For every p holds $p \in \operatorname{dom}(T_{x \leftarrow T'})$ iff $p \in \operatorname{dom} T$ or there exists a node q of T and there exists a node r of T' such that $q \in \operatorname{Leaves} \operatorname{dom} T$ and $T(q) = x$ and $p = q \frown r$,

(ii) for every node p of T such that $p \notin \operatorname{Leaves} \operatorname{dom} T$ or $T(p) \neq x$ holds $T_{x \leftarrow T'}(p) = T(p)$, and

(iii) for every node p of T and for every node q of T' such that $p \in \operatorname{Leaves} \operatorname{dom} T$ and $T(p) = x$ holds $T_{x \leftarrow T'}(p \frown q) = T'(q)$.

Let D be a non empty set, let T, T' be trees decorated by D, and let x be arbitrary. Then $T_{x \leftarrow T'}$ is a tree decorated by D.

We follow a convention: T, T', T_1, T_2 are decorated trees and x, y, z are arbitrary.

One can prove the following proposition

(23) If $x \notin \operatorname{rng} T$ or $x \notin \operatorname{Leaves} T$, then $T_{x \leftarrow T'} = T$.

3. Double Decorated Trees

For simplicity we adopt the following rules: D_1, D_2 are non empty set, T is a tree decorated by D_1 and D_2, F is a non empty set of trees decorated by D_1 and D_2, and F_1 is a non empty set of trees decorated by D_1.

The following propositions are true:

(24) For all D_1, D_2, T holds $\operatorname{dom}(T_1) = \operatorname{dom} T$ and $\operatorname{dom}(T_2) = \operatorname{dom} T$.

(25) (the root tree of $\langle d_1, d_2 \rangle)_1$ = the root tree of d_1 and (the root tree of $\langle d_1, d_2 \rangle)_2$ = the root tree of d_2.

(26) $\langle$the root tree of x, the root tree of $y\rangle$ = the root tree of $\langle x, y \rangle$.

(27) Given D_1, D_2, d_1, d_2, F, F_1, and let p be a finite sequence of elements of F, and let p_1 be a finite sequence of elements of F_1. Suppose $\operatorname{dom} p_1 = \operatorname{dom} p$ and for every i such that $i \in \operatorname{dom} p$ and for every T such that $T = p(i)$ holds $p_1(i) = T_1$. Then $(\langle d_1, d_2 \rangle\text{-tree}(p))_1 = d_1\text{-tree}(p_1)$.

(28) Given D_1, D_2, d_1, d_2, F, F_2, and let p be a finite sequence of elements of F, and let p_2 be a finite sequence of elements of F_2. Suppose $\mathrm{dom}\, p_2 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ and for every T such that $T = p(i)$ holds $p_2(i) = T_2$. Then $(\langle d_1, d_2\rangle\text{-tree}(p))_2 = d_2\text{-tree}(p_2)$.

(29) Given D_1, D_2, d_1, d_2, F and let p be a finite sequence of elements of F. Then there exists a finite sequence p_1 of elements of $\mathrm{Trees}(D_1)$ such that $\mathrm{dom}\, p_1 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ there exists an element T of F such that $T = p(i)$ and $p_1(i) = T_1$ and $(\langle d_1, d_2\rangle\text{-tree}(p))_1 = d_1\text{-tree}(p_1)$.

(30) Given D_1, D_2, d_1, d_2, F and let p be a finite sequence of elements of F. Then there exists a finite sequence p_2 of elements of $\mathrm{Trees}(D_2)$ such that $\mathrm{dom}\, p_2 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ there exists an element T of F such that $T = p(i)$ and $p_2(i) = T_2$ and $(\langle d_1, d_2\rangle\text{-tree}(p))_2 = d_2\text{-tree}(p_2)$.

(31) Given D_1, D_2, d_1, d_2 and let p be a finite sequence of elements of $\mathrm{FinTrees}([\!: D_1, D_2 :\!])$. Then there exists a finite sequence p_1 of elements of $\mathrm{FinTrees}(D_1)$ such that $\mathrm{dom}\, p_1 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ there exists an element T of $\mathrm{FinTrees}([\!: D_1, D_2 :\!])$ such that $T = p(i)$ and $p_1(i) = T_1$ and $(\langle d_1, d_2\rangle\text{-tree}(p))_1 = d_1\text{-tree}(p_1)$.

(32) Given D_1, D_2, d_1, d_2 and let p be a finite sequence of elements of $\mathrm{FinTrees}([\!: D_1, D_2 :\!])$. Then there exists a finite sequence p_2 of elements of $\mathrm{FinTrees}(D_2)$ such that $\mathrm{dom}\, p_2 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ there exists an element T of $\mathrm{FinTrees}([\!: D_1, D_2 :\!])$ such that $T = p(i)$ and $p_2(i) = T_2$ and $(\langle d_1, d_2\rangle\text{-tree}(p))_2 = d_2\text{-tree}(p_2)$.

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Binary Arithmetics

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Summary. Formalizes the basic concepts of binary arithmetic and its related operations. We present the definitions for the following logical operators: 'or' and 'xor' (exclusive or) and include in this article some theorems concerning these operators. We also introduce the concept of an n-bit register. Such registers are used in the definition of binary unsigned arithmetic presented in this article. Theorems on the relationships of such concepts to the operations of natural numbers are also given.

MML Identifier: `BINARITH`.

The notation and terminology used in this paper are introduced in the following papers: [12], [1], [13], [15], [7], [8], [4], [2], [9], [11], [10], [5], [3], [6], and [14].

Let us observe that there exists a natural number which is non empty.

One can prove the following proposition

(1) For all natural numbers i, j holds $+_{\mathbb{N}}(i, j) = i + j$.

Let n be a natural number and let X be a non empty set. A tuple of n and X is an element of X^n.

One can prove the following propositions:

(2) Let i, n be natural numbers, and let D be a non empty set, and let d be an element of D, and let z be a tuple of n and D. If $i \in \operatorname{Seg} n$, then $\pi_i(z ^\frown \langle d \rangle) = \pi_i z$.

(3) Let n be a natural number, and let D be a non empty set, and let d be an element of D, and let z be a tuple of n and D. Then $\pi_{n+1}(z ^\frown \langle d \rangle) = d$.

(4) For every non empty natural number n holds $n \geq 1$.

(5) For all natural numbers i, n such that $i \in \operatorname{Seg} n$ holds i is non empty.

Let x, y be elements of *Boolean*. The functor $x \vee y$ yields an element of *Boolean* and is defined by:

(Def.1) $x \vee y = \neg(\neg x \wedge \neg y)$.

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Let x, y be elements of *Boolean*. The functor $x \oplus y$ yielding an element of *Boolean* is defined by:

(Def.2) $x \oplus y = \neg x \wedge y \vee x \wedge \neg y$.

In the sequel x, y, z will denote elements of *Boolean*.

The following propositions are true:

(6) $x \vee y = y \vee x$.

(7) $x \vee \mathit{false} = x$ and $\mathit{false} \vee x = x$.

(8) $x \vee y = \neg(\neg x \wedge \neg y)$.

(9) $\neg(x \wedge y) = \neg x \vee \neg y$.

(10) $\neg(x \vee y) = \neg x \wedge \neg y$.

(11) $x \oplus y = y \oplus x$.

(12) $x \wedge y = \neg(\neg x \vee \neg y)$.

(13) $\mathit{true} \oplus x = \neg x$ and $x \oplus \mathit{true} = \neg x$.

(14) $\mathit{false} \oplus x = x$ and $x \oplus \mathit{false} = x$.

(15) $x \oplus x = \mathit{false}$.

(16) $x \wedge x = x$.

(17) $x \oplus \neg x = \mathit{true}$ and $\neg x \oplus x = \mathit{true}$.

(18) $x \vee \neg x = \mathit{true}$ and $\neg x \vee x = \mathit{true}$.

(19) $x \vee \mathit{true} = \mathit{true}$ and $\mathit{true} \vee x = \mathit{true}$.

(20) $(x \vee y) \vee z = x \vee (y \vee z)$.

(21) $x \vee x = x$.

(22) $x \wedge (y \vee z) = x \wedge y \vee x \wedge z$.

(23) $x \vee y \wedge z = (x \vee y) \wedge (x \vee z)$.

(24) $x \vee x \wedge y = x$.

(25) $x \wedge (x \vee y) = x$.

(26) $x \vee \neg x \wedge y = x \vee y$.

(27) $x \wedge (\neg x \vee y) = x \wedge y$.

(28) $x \wedge \neg x = \mathit{false}$ and $\neg x \wedge x = \mathit{false}$.

(29) $\mathit{false} \wedge x = \mathit{false}$ and $x \wedge \mathit{false} = \mathit{false}$.

(30) $z \wedge x \wedge y = x \wedge y \wedge z$.

(31) $z \wedge y \wedge x = x \wedge y \wedge z$.

(32) $x \wedge z \wedge y = x \wedge y \wedge z$.

(33) $\mathit{true} \oplus \mathit{false} = \mathit{true}$ and $\mathit{false} \oplus \mathit{true} = \mathit{true}$.

(34) $x \oplus y \oplus z = x \oplus y \oplus z$.

(35) $x \oplus \neg x \wedge y = x \vee y$.

(36) $x \vee x \oplus y = x \vee y$.

(37) $x \vee \neg x \oplus y = x \vee \neg y$.

(38) $x \wedge y \oplus z = x \wedge y \oplus x \wedge z$.

In the sequel i, j, k will be natural numbers.

Let us consider i, j. The functor $i -' j$ yields a natural number and is defined as follows:

(Def.3) (i) $i -' j = i - j$ if $i - j \geq 0$,

(ii) $i -' j = 0$, otherwise.

Next we state the proposition

(39) $(i + j) -' j = i$.

We adopt the following convention: n will denote a non empty natural number and x, y, z, z_1, z_2 will denote tuples of n and *Boolean*.

Let us consider n, x. The functor $\neg x$ yields a tuple of n and *Boolean* and is defined as follows:

(Def.4) For every i such that $i \in \mathrm{Seg}\, n$ holds $\pi_i \neg x = \neg \pi_i x$.

Let us consider y. The functor $\mathrm{carry}(x, y)$ yielding a tuple of n and *Boolean* is defined as follows:

(Def.5) $\pi_1 \mathrm{carry}(x, y) = \mathit{false}$ and for every i such that $1 \leq i$ and $i < n$ holds $\pi_{i+1} \mathrm{carry}(x, y) = \pi_i x \wedge \pi_i y \vee \pi_i x \wedge \pi_i \mathrm{carry}(x, y) \vee \pi_i y \wedge \pi_i \mathrm{carry}(x, y)$.

Let us consider n, x. The functor $\mathrm{Binary}(x)$ yielding a tuple of n and $\mathbb{N}$ is defined by:

(Def.6) For every i such that $i \in \mathrm{Seg}\, n$ holds $\pi_i \mathrm{Binary}(x) = (\pi_i x = \mathit{false} \to 0, \text{the } i -' 1\text{-th power of } 2)$.

Let us consider n, x. The functor $\mathrm{Absval}(x)$ yielding a natural number is defined by:

(Def.7) $\mathrm{Absval}(x) = +_{\mathbb{N}} \circledast \mathrm{Binary}(x)$.

Let us consider n, x, y. The functor $x + y$ yielding a tuple of n and *Boolean* is defined by:

(Def.8) For every i such that $i \in \mathrm{Seg}\, n$ holds $\pi_i(x+y) = \pi_i x \oplus \pi_i y \oplus \pi_i \mathrm{carry}(x, y)$.

Let us consider n, z_1, z_2. The functor $\mathrm{add\_ovfl}(z_1, z_2)$ yielding an element of *Boolean* is defined by:

(Def.9) $\mathrm{add\_ovfl}(z_1, z_2) = \pi_n z_1 \wedge \pi_n z_2 \vee \pi_n z_1 \wedge \pi_n \mathrm{carry}(z_1, z_2) \vee \pi_n z_2 \wedge \pi_n \mathrm{carry}(z_1, z_2)$.

Let us consider n, z_1, z_2. We say that z_1 and z_2 are summable if and only if:

(Def.10) $\mathrm{add\_ovfl}(z_1, z_2) = \mathit{false}$.

Let us consider n, k. Then $n + k$ is a non empty natural number.

One can prove the following proposition

(40) For every tuple z_1 of 1 and *Boolean* holds $z_1 = \langle \mathit{false} \rangle$ or $z_1 = \langle \mathit{true} \rangle$.

Let n_1 be a non empty natural number, let n_2 be a natural number, let D be a non empty set, let z_1 be a tuple of n_1 and D, and let z_2 be a tuple of n_2 and D. Then $z_1 \frown z_2$ is a tuple of $n_1 + n_2$ and D.

Let D be a non empty set and let d be an element of D. Then $\langle d \rangle$ is a tuple of 1 and D.

The following propositions are true:

(41) Given n, and let z_1, z_2 be tuples of n and *Boolean*, and let d_1, d_2 be elements of *Boolean*, and let i be a natural number. If $i \in \operatorname{Seg} n$, then $\pi_i \operatorname{carry}(z_1 \frown \langle d_1 \rangle, z_2 \frown \langle d_2 \rangle) = \pi_i \operatorname{carry}(z_1, z_2)$.

(42) For every n and for all tuples z_1, z_2 of n and *Boolean* and for all elements d_1, d_2 of *Boolean* holds $\operatorname{add\_ovfl}(z_1, z_2) = \pi_{n+1} \operatorname{carry}(z_1 \frown \langle d_1 \rangle, z_2 \frown \langle d_2 \rangle)$.

(43) For every n and for all tuples z_1, z_2 of n and *Boolean* and for all elements d_1, d_2 of *Boolean* holds $z_1 \frown \langle d_1 \rangle + z_2 \frown \langle d_2 \rangle = (z_1 + z_2) \frown \langle d_1 \oplus d_2 \oplus \operatorname{add\_ovfl}(z_1, z_2) \rangle$.

(44) For every n and for every tuple z of n and *Boolean* and for every element d of *Boolean* holds $\operatorname{Absval}(z \frown \langle d \rangle) = \operatorname{Absval}(z) + (d = \mathit{false} \to 0, \text{the } n\text{-th power of } 2)$.

(45) For every n and for all tuples z_1, z_2 of n and *Boolean* holds $\operatorname{Absval}(z_1 + z_2) + (\operatorname{add\_ovfl}(z_1, z_2) = \mathit{false} \to 0, \text{the } n\text{-th power of } 2) = \operatorname{Absval}(z_1) + \operatorname{Absval}(z_2)$.

(46) For every n and for all tuples z_1, z_2 of n and *Boolean* such that z_1 and z_2 are summable holds $\operatorname{Absval}(z_1 + z_2) = \operatorname{Absval}(z_1) + \operatorname{Absval}(z_2)$.

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Basic Concepts for Petri Nets with Boolean Markings

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Summary. Contains basic concepts for Petri nets with Boolean markings and the firability/firing of single transitions as well as sequences of transitions [7]. The concept of a Boolean marking is introduced as a mapping of a Boolean TRUE/FALSE to each of the places in a place/transition net. This simplifies the conventional definitions of the firability and firing of a transition. One note of caution in this article - the definition of firing a transition does not require that the transition be firable. Therefore, it is advisable to check that transitions ARE firable before firing them.

MML Identifier: BOOLMARK.

The papers [12], [1], [15], [17], [18], [4], [5], [13], [10], [11], [9], [2], [3], [14], [6], [16], and [8] provide the notation and terminology for this paper.

1. Preliminaries

The following four propositions are true:

(1) Let A, B be non empty set, and let f be a function from A into B, and let C be a subset of A, and let v be an element of B. Then $f +\cdot (C \longmapsto v)$ is a function from A into B.

(2) Let X, Y be non empty set, and let A, B be subsets of X, and let f be a function from X into Y. If $f^\circ A \cap f^\circ B = \emptyset$, then $A \cap B = \emptyset$.

(3) For all sets A, B and for every function f and for arbitrary x such that $A \cap B = \emptyset$ holds $(f +\cdot (A \longmapsto x))^\circ B = f^\circ B$.

(4) Let n be a natural number, and let D be a non empty set, and let d be an element of D, and let z be a finite sequence of elements of D. If $\operatorname{len} z = n$, then $\pi_{n+1}(z \frown \langle d \rangle) = d$.

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2. Boolean Marking and Firability/Firing of Transitions

Let P_1 be a place/transition net structure. The functor Bool_marks_of P_1 yielding a non empty set of functions from the places of P_1 to *Boolean* is defined by:

(Def.1) Bool_marks_of $P_1 = Boolean^{\text{the places of } P_1}$.

Let P_1 be a place/transition net structure. A Boolean marking of P_1 is an element of Bool_marks_of P_1.

Let P_1 be a place/transition net structure, let M_0 be a Boolean marking of P_1, and let t be a transition of P_1. We say that t is firable on M_0 if and only if:

(Def.2) $M_0 \circ ({}^*\{t\}) \subseteq \{true\}$.

Let P_1 be a place/transition net structure, let M_0 be a Boolean marking of P_1, and let t be a transition of P_1. The functor Firing(t, M_0) yields a Boolean marking of P_1 and is defined by:

(Def.3) $\text{Firing}(t, M_0) = M_0 +\cdot ({}^*\{t\} \longmapsto false) +\cdot (\{t\}^* \longmapsto true)$.

Let P_1 be a place/transition net structure, let M_0 be a Boolean marking of P_1, and let Q be a finite sequence of elements of the transitions of P_1. We say that Q is firable on M_0 if and only if the conditions (Def.4) are satisfied.

(Def.4) (i) $Q = \varepsilon$, or

(ii) there exists a finite sequence M of elements of Bool_marks_of P_1 such that $\text{len}\, Q = \text{len}\, M$ and $\pi_1 Q$ is firable on M_0 and $\pi_1 M = \text{Firing}(\pi_1 Q, M_0)$ and for every natural number i such that $i < \text{len}\, Q$ and $i > 0$ holds $\pi_{i+1} Q$ is firable on $\pi_i M$ and $\pi_{i+1} M = \text{Firing}(\pi_{i+1} Q, \pi_i M)$.

Let P_1 be a place/transition net structure, let M_0 be a Boolean marking of P_1, and let Q be a finite sequence of elements of the transitions of P_1. The functor Firing(Q, M_0) yielding a Boolean marking of P_1 is defined as follows:

(Def.5) (i) $\text{Firing}(Q, M_0) = M_0$ if $Q = \varepsilon$,

(ii) there exists a finite sequence M of elements of Bool_marks_of P_1 such that $\text{len}\, Q = \text{len}\, M$ and $\text{Firing}(Q, M_0) = \pi_{\text{len}\, M} M$ and $\pi_1 M = \text{Firing}(\pi_1 Q, M_0)$ and for every natural number i such that $i < \text{len}\, Q$ and $i > 0$ holds $\pi_{i+1} M = \text{Firing}(\pi_{i+1} Q, \pi_i M)$, otherwise.

One can prove the following propositions:

(5) For every non empty set A and for arbitrary y and for every function f holds $(f +\cdot (A \longmapsto y)) \circ A = \{y\}$.

(6) Let P_1 be a place/transition net structure, and let M_0 be a Boolean marking of P_1, and let t be a transition of P_1, and let s be a place of P_1. If $s \in \{t\}^*$, then $(\text{Firing}(t, M_0))(s) = true$.

(7) Let P_1 be a place/transition net structure and let S_1 be a non empty set of places of P_1. Then S_1 is deadlock-like if and only if for every Boolean marking M_0 of P_1 such that $M_0 \circ S_1 = \{false\}$ and for every transition t of P_1 such that t is firable on M_0 holds $(\text{Firing}(t, M_0)) \circ S_1 = \{false\}$.

(8) Let D be a non empty set, and let Q_0, Q_1 be finite sequences of elements of D, and let i be a natural number. If $1 \leq i$ and $i \leq \operatorname{len} Q_0$, then $\pi_i(Q_0 \frown Q_1) = \pi_i Q_0$.

(9) Let D be a non empty set, and let Q_0, Q_1 be finite sequences of elements of D, and let i be a natural number. If $1 \leq i$ and $i \leq \operatorname{len} Q_1$, then $\pi_{\operatorname{len} Q_0 + i}(Q_0 \frown Q_1) = \pi_i Q_1$.

(10) Let P_1 be a place/transition net structure, and let M_0 be a Boolean marking of P_1, and let Q_0, Q_1 be finite sequences of elements of the transitions of P_1. Then $\text{Firing}(Q_0 \frown Q_1, M_0) = \text{Firing}(Q_1, \text{Firing}(Q_0, M_0))$.

(11) Let P_1 be a place/transition net structure, and let M_0 be a Boolean marking of P_1, and let Q_0, Q_1 be finite sequences of elements of the transitions of P_1. If $Q_0 \frown Q_1$ is firable on M_0, then Q_1 is firable on $\text{Firing}(Q_0, M_0)$ and Q_0 is firable on M_0.

(12) Let P_1 be a place/transition net structure, and let M_0 be a Boolean marking of P_1, and let t be a transition of P_1. Then t is firable on M_0 if and only if $\langle t \rangle$ is firable on M_0.

(13) Let P_1 be a place/transition net structure, and let M_0 be a Boolean marking of P_1, and let t be a transition of P_1. Then $\text{Firing}(t, M_0) = \text{Firing}(\langle t \rangle, M_0)$.

(14) Let P_1 be a place/transition net structure and let S_1 be a non empty set of places of P_1. Then S_1 is deadlock-like if and only if for every Boolean marking M_0 of P_1 such that $M_0 \circ S_1 = \{\mathit{false}\}$ and for every finite sequence Q of elements of the transitions of P_1 such that Q is firable on M_0 holds $(\text{Firing}(Q, M_0)) \circ S_1 = \{\mathit{false}\}$.

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On Defining Functions on Trees [1]

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Summary. The continuation of the sequence of articles on trees (see [3,5,7,4]) and on context-free grammars ([15]). We define the set of complete parse trees for a given context-free grammar. Next we define the scheme of induction for the set and the scheme of defining functions by induction on the set. For each symbol of a context-free grammar we define the terminal, the pretraversal, and the posttraversal languages. The introduced terminology is tested on the example of Peano naturals.

MML Identifier: DTCONSTR.

The terminology and notation used in this paper are introduced in the following articles: [18], [2], [21], [12], [13], [9], [1], [14], [8], [11], [16], [19], [6], [17], [10], [20], [15], [3], [5], [7], and [4].

1. Preliminaries

The following propositions are true:

(1) For every non empty set D holds every finite sequence of elements of $\mathrm{FinTrees}(D)$ is a finite sequence of elements of $\mathrm{Trees}(D)$.

(2) For arbitrary x, y and for every finite sequence p of elements of x such that $y \in \mathrm{dom}\, p$ or $y \in \mathrm{Seg}\,\mathrm{len}\, p$ holds $p(y) \in x$.

Let X be a set. Observe that every element of X^* is function-like.

Let X be a set. Note that every element of X^* is finite sequence-like.

Let D be a set and let p, q be elements of D^*. Then $p \frown q$ is an element of D^*.

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Let D be a non empty set and let t be an element of FinTrees(D). Then dom t is a finite tree.

Let D be a non empty set and let T be a set of trees decorated by D. One can verify that every finite sequence of elements of T is decorated tree yielding.

Let D be a non empty set, let F be a non empty set of trees decorated by D, and let T_1 be a non empty subset of F. We see that the element of T_1 is an element of F.

Let p be a finite sequence. Let us assume that p is decorated tree yielding. The roots of p constitute finite sequences and is defined by the conditions (Def.1).

(Def.1) (i) dom (the roots of p) = dom p, and

(ii) for every natural number i such that $i \in \operatorname{dom} p$ there exists a decorated tree T such that $T = p(i)$ and (the roots of p)(i) = $T(\varepsilon)$.

Let D be a non empty set, let T be a set of trees decorated by D, and let p be a finite sequence of elements of T. Then the roots of p is a finite sequence of elements of D.

One can prove the following propositions:

(3) The roots of $\varepsilon = \varepsilon$.

(4) For every decorated tree T holds the roots of $\langle T \rangle = \langle T(\varepsilon) \rangle$.

(5) Let D be a non empty set, and let F be a subset of FinTrees(D), and let p be a finite sequence of elements of F. Suppose len (the roots of p) = 1. Then there exists an element x of FinTrees(D) such that $p = \langle x \rangle$ and $x \in F$.

(6) For all decorated trees T_2, T_3 holds the roots of $\langle T_2, T_3 \rangle = \langle T_2(\varepsilon), T_3(\varepsilon) \rangle$.

Let f be a function. The functor pr1(f) yields a function and is defined by:

(Def.2) dom pr1(f) = dom f and for arbitrary x such that $x \in \operatorname{dom} f$ holds $\mathrm{pr1}(f)(x) = f(x)_{\mathbf{1}}$.

The functor pr2(f) yielding a function is defined by:

(Def.3) dom pr2(f) = dom f and for arbitrary x such that $x \in \operatorname{dom} f$ holds $\mathrm{pr2}(f)(x) = f(x)_{\mathbf{2}}$.

Let X, Y be sets and let f be a finite sequence of elements of $[: X, Y :]$. Then pr1(f) is a finite sequence of elements of X. Then pr2(f) is a finite sequence of elements of Y.

One can prove the following proposition

(7) $\mathrm{pr1}(\varepsilon) = \varepsilon$ and $\mathrm{pr2}(\varepsilon) = \varepsilon$.

The scheme *MonoSetSeq* concerns a function $\mathcal{A}$, a set $\mathcal{B}$, and a binary functor $\mathcal{F}$ yielding a set, and states that:

For all natural numbers k, s holds $\mathcal{A}(k) \subseteq \mathcal{A}(k+s)$

provided the parameters meet the following requirement:

- For every natural number n and for arbitrary x such that $x = \mathcal{A}(n)$ holds $\mathcal{A}(n+1) = x \cup \mathcal{F}(n, x)$.

2. The set of parse trees

Now we present two schemes. The scheme *DTConstrStrEx* concerns a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a strict tree construction structure G such that
(i) the carrier of $G = \mathcal{A}$, and
(ii) for every symbol x of G and for every finite sequence p of elements of the carrier of G holds $x \Rightarrow p$ iff $\mathcal{P}[x,p]$

for all values of the parameters.

The scheme *DTConstrStrUniq* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

Let G_1, G_2 be strict tree construction structure. Suppose that
(i) the carrier of $G_1 = \mathcal{A}$,
(ii) for every symbol x of G_1 and for every finite sequence p of elements of the carrier of G_1 holds $x \Rightarrow p$ iff $\mathcal{P}[x,p]$,
(iii) the carrier of $G_2 = \mathcal{A}$, and
(iv) for every symbol x of G_2 and for every finite sequence p of elements of the carrier of G_2 holds $x \Rightarrow p$ iff $\mathcal{P}[x,p]$.

Then $G_1 = G_2$

for all values of the parameters.

Next we state the proposition

(8) For every tree construction structure G holds (the terminals of G)$\cap$(the nonterminals of G) $= \emptyset$.

Now we present four schemes. The scheme *DTCMin* concerns a function $\mathcal{A}$, a tree construction structure $\mathcal{B}$, a non empty set $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

There exists a subset X of FinTrees([: the carrier of $\mathcal{B}$, $\mathcal{C}$:]) such that
(i) $X = \bigcup \mathcal{A}$,
(ii) for every symbol d of $\mathcal{B}$ such that $d \in$ the terminals of $\mathcal{B}$ holds the root tree of $\langle d, \mathcal{F}(d)\rangle \in X$,
(iii) for every symbol o of $\mathcal{B}$ and for every finite sequence p of elements of X such that $o \Rightarrow$ pr1(the roots of p) and for arbitrary s, v such that $s =$ pr1(the roots of p) and $v =$ pr2(the roots of p) holds $\langle o, \mathcal{G}(o,s,v)\rangle$-tree($p$) $\in X$, and
(iv) for every subset F of FinTrees([: the carrier of $\mathcal{B}$, $\mathcal{C}$:]) such that for every symbol d of $\mathcal{B}$ such that $d \in$ the terminals of $\mathcal{B}$ holds the root tree of $\langle d, \mathcal{F}(d)\rangle \in F$ and for every symbol o of $\mathcal{B}$ and for every finite sequence p of elements of F such that $o \Rightarrow$ pr1(the roots of p) holds $\langle o, \mathcal{G}(o,$ pr1(the roots of p), pr2(the roots of p))$\rangle$-tree(p) $\in F$ holds $X \subseteq F$

provided the following conditions are satisfied:

- $\operatorname{dom} \mathcal{A} = \mathbb{N}$,
- $\mathcal{A}(0) = \{$the root tree of $\langle t, d\rangle$: t ranges over symbols of $\mathcal{B}$, d ranges over elements of $\mathcal{C}$, $t \in$ the terminals of $\mathcal{B} \wedge d = \mathcal{F}(t) \vee t \Rightarrow \varepsilon \wedge d = \mathcal{G}(t, \varepsilon, \varepsilon)\}$,
- Let n be a natural number and let x be arbitrary. Suppose $x = \mathcal{A}(n)$. Then $\mathcal{A}(n+1) = x \cup \{\langle o, \mathcal{G}(o, \text{pr1(the roots of } p), \text{pr2(the roots of } p))\rangle\text{-tree}(p) : o$ ranges over symbols of $\mathcal{B}$, p ranges over elements of x^*, $\exists_q\ p = q \ \wedge\ o \Rightarrow$ pr1(the roots of $q)\}$.

The scheme *DTCSymbols* deals with a function $\mathcal{A}$, a tree construction structure $\mathcal{B}$, a non empty set $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

There exists a subset X_1 of FinTrees(the carrier of $\mathcal{B}$) such that

(i) $X_1 = \{t_1 : t$ ranges over elements of FinTrees([: the carrier of $\mathcal{B}$, $\mathcal{C}$:]), $t \in \bigcup \mathcal{A}\}$,

(ii) for every symbol d of $\mathcal{B}$ such that $d \in$ the terminals of $\mathcal{B}$ holds the root tree of $d \in X_1$,

(iii) for every symbol o of $\mathcal{B}$ and for every finite sequence p of elements of X_1 such that $o \Rightarrow$ the roots of p holds o-tree$(p) \in X_1$, and

(iv) for every subset F of FinTrees(the carrier of $\mathcal{B}$) such that for every symbol d of $\mathcal{B}$ such that $d \in$ the terminals of $\mathcal{B}$ holds the root tree of $d \in F$ and for every symbol o of $\mathcal{B}$ and for every finite sequence p of elements of F such that $o \Rightarrow$ the roots of p holds o-tree$(p) \in F$ holds $X_1 \subseteq F$

provided the parameters meet the following requirements:

- $\operatorname{dom} \mathcal{A} = \mathbb{N}$,
- $\mathcal{A}(0) = \{$the root tree of $\langle t, d\rangle$: t ranges over symbols of $\mathcal{B}$, d ranges over elements of $\mathcal{C}$, $t \in$ the terminals of $\mathcal{B} \wedge d = \mathcal{F}(t) \vee t \Rightarrow \varepsilon \wedge d = \mathcal{G}(t, \varepsilon, \varepsilon)\}$,
- Let n be a natural number and let x be arbitrary. Suppose $x = \mathcal{A}(n)$. Then $\mathcal{A}(n+1) = x \cup \{\langle o, \mathcal{G}(o, \text{pr1(the roots of } p), \text{pr2(the roots of } p))\rangle\text{-tree}(p) : o$ ranges over symbols of $\mathcal{B}$, p ranges over elements of x^*, $\exists_q\ p = q \ \wedge\ o \Rightarrow$ pr1(the roots of $q)\}$.

The scheme *DTCHeight* concerns a function $\mathcal{A}$, a tree construction structure $\mathcal{B}$, a non empty set $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

Let n be a natural number and let d_1 be an element of FinTrees([: the carrier of $\mathcal{B}$, $\mathcal{C}$:]). If $d_1 \in \bigcup \mathcal{A}$, then $d_1 \in \mathcal{A}(n)$ iff height dom $d_1 \leq n$

provided the parameters meet the following conditions:

- $\operatorname{dom} \mathcal{A} = \mathbb{N}$,
- $\mathcal{A}(0) = \{$the root tree of $\langle t, d\rangle$: t ranges over symbols of $\mathcal{B}$, d ranges over elements of $\mathcal{C}$, $t \in$ the terminals of $\mathcal{B} \wedge d = \mathcal{F}(t) \vee t \Rightarrow \varepsilon \wedge d = \mathcal{G}(t, \varepsilon, \varepsilon)\}$,
- Let n be a natural number and let x be arbitrary. Suppose $x = \mathcal{A}(n)$. Then $\mathcal{A}(n+1) = x \cup \{\langle o, \mathcal{G}(o, \text{pr1(the roots of } p), \text{pr2(the

roots of p))⟩-tree(p) : o ranges over symbols of $\mathcal{B}$, p ranges over elements of x^*, $\exists_q\ p = q\ \wedge\ o \Rightarrow$ pr1(the roots of q)}.

The scheme *DTCUniq* concerns a function $\mathcal{A}$, a tree construction structure $\mathcal{B}$, a non empty set $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

For all trees d_2, d_3 decorated by [:the carrier of $\mathcal{B}$, $\mathcal{C}$:] such that $d_2 \in \bigcup\mathcal{A}$ and $d_3 \in \bigcup\mathcal{A}$ and $(d_2)_1 = (d_3)_1$ holds $d_2 = d_3$

provided the following conditions are satisfied:

- dom $\mathcal{A} = \mathbb{N}$,
- $\mathcal{A}(0) = \{$the root tree of ⟨t, d⟩: t ranges over symbols of $\mathcal{B}$, d ranges over elements of $\mathcal{C}$, $t \in$ the terminals of $\mathcal{B} \wedge d = \mathcal{F}(t) \vee t \Rightarrow \varepsilon \wedge d = \mathcal{G}(t, \varepsilon, \varepsilon)\}$,
- Let n be a natural number and let x be arbitrary. Suppose $x = \mathcal{A}(n)$. Then $\mathcal{A}(n+1) = x \cup \{$⟨$o$, $\mathcal{G}(o$, pr1(the roots of p), pr2(the roots of p))⟩-tree(p) : o ranges over symbols of $\mathcal{B}$, p ranges over elements of x^*, $\exists_q\ p = q\ \wedge\ o \Rightarrow$ pr1(the roots of q)}.

Let G be a tree construction structure. The functor TS(G) yields a subset of FinTrees(the carrier of G) and is defined by the conditions (Def.4).

(Def.4) (i) For every symbol d of G such that $d \in$ the terminals of G holds the root tree of $d \in$ TS(G),

(ii) for every symbol o of G and for every finite sequence p of elements of TS(G) such that $o \Rightarrow$ the roots of p holds o-tree(p) $\in$ TS(G), and

(iii) for every subset F of FinTrees(the carrier of G) such that for every symbol d of G such that $d \in$ the terminals of G holds the root tree of $d \in F$ and for every symbol o of G and for every finite sequence p of elements of F such that $o \Rightarrow$ the roots of p holds o-tree(p) $\in F$ holds TS(G) $\subseteq F$.

Now we present three schemes. The scheme *DTConstrInd* concerns a tree construction structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every tree t decorated by the carrier of $\mathcal{A}$ such that $t \in$ TS($\mathcal{A}$) holds $\mathcal{P}[t]$

provided the parameters meet the following requirements:

- For every symbol s of $\mathcal{A}$ such that $s \in$ the terminals of $\mathcal{A}$ holds $\mathcal{P}$[the root tree of s],
- Let n_1 be a symbol of $\mathcal{A}$ and let t_1 be a finite sequence of elements of TS($\mathcal{A}$). Suppose $n_1 \Rightarrow$ the roots of t_1 and for every tree t decorated by the carrier of $\mathcal{A}$ such that $t \in \operatorname{rng} t_1$ holds $\mathcal{P}[t]$. Then $\mathcal{P}[n_1$-tree$(t_1)]$.

The scheme *DTConstrIndDef* concerns a tree construction structure $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and states that:

There exists a function f from TS($\mathcal{A}$) into $\mathcal{B}$ such that

(i) for every symbol t of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds f(the root tree of t) $= \mathcal{F}(t)$, and

(ii) for every symbol n_1 of $\mathcal{A}$ and for every finite sequence t_1 of elements of TS($\mathcal{A}$) and for every finite sequence r_1 such that $r_1 =$ the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of $\mathcal{B}$ such that $x = f \cdot t_1$ holds $f(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, r_1, x)$

for all values of the parameters.

The scheme *DTConstrUniqDef* deals with a tree construction structure $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and functions $\mathcal{C}$, $\mathcal{D}$ from TS($\mathcal{A}$) into $\mathcal{B}$, and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters satisfy the following conditions:

- (i) For every symbol t of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds $\mathcal{C}$(the root tree of t) $= \mathcal{F}(t)$, and
  (ii) for every symbol n_1 of $\mathcal{A}$ and for every finite sequence t_1 of elements of TS($\mathcal{A}$) and for every finite sequence r_1 such that $r_1 =$ the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of $\mathcal{B}$ such that $x = \mathcal{C} \cdot t_1$ holds $\mathcal{C}(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, r_1, x)$,
- (i) For every symbol t of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds $\mathcal{D}$(the root tree of t) $= \mathcal{F}(t)$, and
  (ii) for every symbol n_1 of $\mathcal{A}$ and for every finite sequence t_1 of elements of TS($\mathcal{A}$) and for every finite sequence r_1 such that $r_1 =$ the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of $\mathcal{B}$ such that $x = \mathcal{D} \cdot t_1$ holds $\mathcal{D}(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, r_1, x)$.

3. An example: Peano naturals

The strict tree construction structure $\mathbb{N}_{\text{Peano}}$ is defined by the conditions (Def.5).

(Def.5) (i) The carrier of $\mathbb{N}_{\text{Peano}} = \{0, 1\}$, and
(ii) for every symbol x of $\mathbb{N}_{\text{Peano}}$ and for every finite sequence y of elements of the carrier of $\mathbb{N}_{\text{Peano}}$ holds $x \Rightarrow y$ iff $x = 1$ but $y = \langle 0 \rangle$ or $y = \langle 1 \rangle$.

4. Properties of parse trees

Let G be a tree construction structure. We say that G has terminals if and only if:

(Def.6) The terminals of $G \neq \emptyset$.

We say that G has nonterminals if and only if:

(Def.7) The nonterminals of $G \neq \emptyset$.

We say that G has useful nonterminals if and only if the condition (Def.8) is satisfied.

(Def.8) Let n_1 be a symbol of G. Suppose $n_1 \in$ the nonterminals of G. Then there exists a finite sequence p of elements of TS(G) such that $n_1 \Rightarrow$ the roots of p.

Let us note that there exists a tree construction structure which is strict and has terminals, nonterminals, and useful nonterminals.

Let G be a tree construction structure with terminals. Then the terminals of G is a non empty subset of the carrier of G. Then TS(G) is a non empty subset of FinTrees(the carrier of G).

Let G be a tree construction structure with useful nonterminals. Then TS(G) is a non empty subset of FinTrees(the carrier of G).

Let G be a tree construction structure with nonterminals. Then the nonterminals of G is a non empty subset of the carrier of G.

Let G be a tree construction structure with terminals. A terminal of G is an element of the terminals of G.

Let G be a tree construction structure with nonterminals. A nonterminal of G is an element of the nonterminals of G.

Let G be a tree construction structure with nonterminals and useful nonterminals and let n_1 be a nonterminal of G. A finite sequence of elements of TS(G) is called a subtree sequence joinable by n_1 if:

(Def.9) $n_1 \Rightarrow$ the roots of it.

Let G be a tree construction structure with terminals and let t be a terminal of G. Then the root tree of t is an element of TS(G).

Let G be a tree construction structure with nonterminals and useful nonterminals, let n_1 be a nonterminal of G, and let p be a subtree sequence joinable by n_1. Then n_1-tree(p) is an element of TS(G).

One can prove the following two propositions:

(9) Let G be a tree construction structure with terminals, and let t_2 be an element of TS(G), and let s be a terminal of G. If $t_2(\varepsilon) = s$, then $t_2 =$ the root tree of s.

(10) Let G be a tree construction structure with terminals and nonterminals, and let t_2 be an element of TS(G), and let n_1 be a nonterminal of G. Suppose $t_2(\varepsilon) = n_1$. Then there exists a finite sequence t_1 of elements of TS(G) such that $t_2 = n_1$-tree(t_1) and $n_1 \Rightarrow$ the roots of t_1.

5. The Example Continued

$\mathbb{N}_{\mathrm{Peano}}$ is a strict tree construction structure with terminals, nonterminals, and useful nonterminals.

Let n_1 be a nonterminal of $\mathbb{N}_{\mathrm{Peano}}$ and let t be an element of TS($\mathbb{N}_{\mathrm{Peano}}$). Then n_1-tree(t) is an element of TS($\mathbb{N}_{\mathrm{Peano}}$).

Let x be a finite sequence of elements of $\mathbb{N}$. Let us assume that $x \neq \varepsilon$. The functor $(x)(_1{+}1)$ yielding a natural number is defined as follows:

(Def.10) There exists a natural number n such that $(x)(_1{+}1) = n+1$ and $x(1) = n$.

The function $\mathbb{N}_{\mathrm{Peano}} \to \mathbb{N}$ from $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ into $\mathbb{N}$ is defined by the conditions (Def.11).

(Def.11) (i) For every symbol t of $\mathbb{N}_{\mathrm{Peano}}$ such that $t \in$ the terminals of $\mathbb{N}_{\mathrm{Peano}}$ holds $(\mathbb{N}_{\mathrm{Peano}} \to \mathbb{N})$(the root tree of t) $= 0$, and

(ii) for every symbol n_1 of $\mathbb{N}_{\mathrm{Peano}}$ and for every finite sequence t_1 of elements of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ and for every finite sequence r_1 such that $r_1 =$ the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of $\mathbb{N}$ such that $x = (\mathbb{N}_{\mathrm{Peano}} \to \mathbb{N}) \cdot t_1$ holds $(\mathbb{N}_{\mathrm{Peano}} \to \mathbb{N})(n_1\text{-tree}(t_1)) = (x)(_1{+}1)$.

Let x be an element of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$. The functor $\mathrm{succ}(x)$ yielding an element of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ is defined as follows:

(Def.12) $\mathrm{succ}(x) = 1\text{-tree}(\langle x \rangle)$.

The function $\mathbb{N} \to \mathbb{N}_{\mathrm{Peano}}$ from $\mathbb{N}$ into $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ is defined by the conditions (Def.13).

(Def.13) (i) $(\mathbb{N} \to \mathbb{N}_{\mathrm{Peano}})(0) =$ the root tree of 0, and

(ii) for every natural number n and for every element x of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ such that $x = (\mathbb{N} \to \mathbb{N}_{\mathrm{Peano}})(n)$ holds $(\mathbb{N} \to \mathbb{N}_{\mathrm{Peano}})(n+1) = \mathrm{succ}(x)$.

One can prove the following propositions:

(11) For every element p_1 of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ holds $p_1 = (\mathbb{N} \to \mathbb{N}_{\mathrm{Peano}})((\mathbb{N}_{\mathrm{Peano}} \to \mathbb{N})(p_1))$.

(12) For every natural number n holds $n = (\mathbb{N}_{\mathrm{Peano}} \to \mathbb{N})((\mathbb{N} \to \mathbb{N}_{\mathrm{Peano}})(n))$.

6. Tree traversals and terminal language

Let D be a set and let F be a finite sequence of elements of D^*. The functor $\mathrm{Flat}(F)$ yields an element of D^* and is defined as follows:

(Def.14) There exists a binary operation g on D^* such that for all elements p, q of D^* holds $g(p, q) = p ^\frown q$ and $\mathrm{Flat}(F) = g \odot F$.

Next we state the proposition

(13) For every set D and for every element d of D^* holds $\mathrm{Flat}(\langle d \rangle) = d$.

Let G be a tree construction structure and let t_2 be a tree decorated by the carrier of G. Let us assume that $t_2 \in \mathrm{TS}(G)$. The terminals of t_2 is a finite sequence of elements of the terminals of G and is defined by the condition (Def.15).

(Def.15) There exists a function f from $\mathrm{TS}(G)$ into (the terminals of G)* such that

(i) the terminals of $t_2 = f(t_2)$,

(ii) for every symbol t of G such that $t \in$ the terminals of G holds f(the root tree of t) $= \langle t \rangle$, and

(iii) for every symbol n_1 of G and for every finite sequence t_1 of elements of TS(G) and for every finite sequence r_1 such that $r_1 =$ the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of (the terminals of G)* such that $x = f \cdot t_1$ holds $f(n_1\text{-tree}(t_1)) = \text{Flat}(x)$.

The pretraversal string of t_2 is a finite sequence of elements of the carrier of G and is defined by the condition (Def.16).

(Def.16) There exists a function f from TS(G) into (the carrier of G)* such that

(i) the pretraversal string of $t_2 = f(t_2)$,

(ii) for every symbol t of G such that $t \in$ the terminals of G holds f(the root tree of t) $= \langle t \rangle$, and

(iii) for every symbol n_1 of G and for every finite sequence t_1 of elements of TS(G) and for every finite sequence r_1 such that $r_1 =$ the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of (the carrier of G)* such that $x = f \cdot t_1$ holds $f(n_1\text{-tree}(t_1)) = \langle n_1 \rangle ^\frown \text{Flat}(x)$.

The posttraversal string of t_2 is a finite sequence of elements of the carrier of G and is defined by the condition (Def.17).

(Def.17) There exists a function f from TS(G) into (the carrier of G)* such that

(i) the posttraversal string of $t_2 = f(t_2)$,

(ii) for every symbol t of G such that $t \in$ the terminals of G holds f(the root tree of t) $= \langle t \rangle$, and

(iii) for every symbol n_1 of G and for every finite sequence t_1 of elements of TS(G) and for every finite sequence r_1 such that $r_1 =$ the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of (the carrier of G)* such that $x = f \cdot t_1$ holds $f(n_1\text{-tree}(t_1)) = \text{Flat}(x) ^\frown \langle n_1 \rangle$.

Let G be a tree construction structure with nonterminals and let n_1 be a symbol of G. The language derivable from n_1 is a subset of (the terminals of G)* and is defined by the condition (Def.18).

(Def.18) The language derivable from $n_1 = \{$the terminals of t_2: t_2 ranges over elements of FinTrees(the carrier of G), $t_2 \in$ TS(G) $\wedge$ $t_2(\varepsilon) = n_1\}$.

The language of pretraversals derivable from n_1 is a subset of (the carrier of G)* and is defined by the condition (Def.19).

(Def.19) The language of pretraversals derivable from $n_1 = \{$the pretraversal string of t_2: t_2 ranges over elements of FinTrees(the carrier of G), $t_2 \in$ TS(G) $\wedge$ $t_2(\varepsilon) = n_1\}$.

The language of posttraversals derivable from n_1 is a subset of (the carrier of G)* and is defined by the condition (Def.20).

(Def.20) The language of posttraversals derivable from $n_1 = \{$the posttraversal string of t_2: t_2 ranges over elements of FinTrees(the carrier of G), $t_2 \in$ TS(G) $\wedge$ $t_2(\varepsilon) = n_1\}$.

One can prove the following propositions:

(14) For every tree t decorated by the carrier of $\mathbb{N}_{\mathrm{Peano}}$ such that $t \in \mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ holds the terminals of $t = \langle 0 \rangle$.

(15) For every symbol n_1 of $\mathbb{N}_{\mathrm{Peano}}$ holds the language derivable from $n_1 = \{\langle 0 \rangle\}$.

(16) For every element t of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ holds the pretraversal string of $t = (\mathrm{height}\,\mathrm{dom}\, t \longmapsto 1) ^\frown \langle 0 \rangle$.

(17) Let n_1 be a symbol of $\mathbb{N}_{\mathrm{Peano}}$. Then
   (i) if $n_1 = 0$, then the language of pretraversals derivable from $n_1 = \{\langle 0 \rangle\}$, and
   (ii) if $n_1 = 1$, then the language of pretraversals derivable from $n_1 = \{(n \longmapsto 1) ^\frown \langle 0 \rangle : n$ ranges over natural numbers, $n \neq 0\}$.

(18) For every element t of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ holds the posttraversal string of $t = \langle 0 \rangle ^\frown (\mathrm{height}\,\mathrm{dom}\, t \longmapsto 1)$.

(19) Let n_1 be a symbol of $\mathbb{N}_{\mathrm{Peano}}$. Then
   (i) if $n_1 = 0$, then the language of posttraversals derivable from $n_1 = \{\langle 0 \rangle\}$, and
   (ii) if $n_1 = 1$, then the language of posttraversals derivable from $n_1 = \{\langle 0 \rangle ^\frown (n \longmapsto 1) : n$ ranges over natural numbers, $n \neq 0\}$.

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Product of Family of Universal Algebras

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Summary. The product of two algebras, trivial algebra determined by an empty set and product of a family of algebras are defined. Some basic properties are shown.

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The terminology and notation used in this paper have been introduced in the following articles: [14], [6], [3], [7], [11], [15], [12], [9], [5], [8], [1], [2], [10], [4], and [13].

1. Product of Two Algebras

The following proposition is true

(1) For all non-empty set D_1, D_2 and for all natural numbers n, m such that ${D_1}^n = {D_2}^m$ holds $n = m$.

For simplicity we follow a convention: U_1, U_2, U_3 denote universal algebras, k, m, i denote natural numbers, z is arbitrary, and h_1, h_2 denote finite sequences of elements of $[: A, B :]$.

Let us consider A, B and let us consider h_1. The functor $\pi_1(h_1)$ yielding a finite sequence of elements of A is defined as follows:

(Def.1) $\operatorname{len} \pi_1(h_1) = \operatorname{len} h_1$ and for every n such that $n \in \operatorname{dom} \pi_1(h_1)$ holds $(\pi_1(h_1))(n) = h_1(n)_1$.

The functor $\pi_2(h_1)$ yielding a finite sequence of elements of B is defined as follows:

(Def.2) $\operatorname{len} \pi_2(h_1) = \operatorname{len} h_1$ and for every n such that $n \in \operatorname{dom} \pi_2(h_1)$ holds $(\pi_2(h_1))(n) = h_1(n)_2$.

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Let us consider A, B, let f_1 be a homogeneous quasi total non-empty partial function from A^* to A, and let f_2 be a homogeneous quasi total non-empty partial function from B^* to B. Let us assume that arity f_1 = arity f_2. The functor $]\!] f_1, f_2 [\![$ yielding a homogeneous quasi total non-empty partial function from $[\!: A, B :\!]^*$ to $[\!: A, B :\!]$ is defined by the conditions (Def.3).

(Def.3) (i) $\mathrm{dom}]\!] f_1, f_2 [\![= [\!: A, B :\!]^{\mathrm{arity} f_1}$, and

(ii) for every finite sequence h of elements of $[\!: A, B :\!]$ such that $h \in \mathrm{dom}]\!] f_1, f_2 [\![$ holds $]\!] f_1, f_2 [\![(h) = \langle f_1(\pi_1(h)), f_2(\pi_2(h)) \rangle$.

In the sequel h_1 will denote a homogeneous quasi total non-empty partial function from (the carrier of U_1)* to the carrier of U_1.

Let us consider U_1, U_2. Let us assume that U_1 and U_2 are similar. The functor $\mathrm{Opers}(U_1, U_2)$ yielding a finite sequence of elements of $[\!:$ the carrier of U_1, the carrier of $U_2 :\!]^* \dot{\rightarrow} [\!:$ the carrier of U_1, the carrier of $U_2 :\!]$ is defined as follows:

(Def.4) len $\mathrm{Opers}(U_1, U_2)$ = len Opers U_1 and for every n such that $n \in$ dom $\mathrm{Opers}(U_1, U_2)$ and for all h_1, h_2 such that $h_1 = (\mathrm{Opers}\, U_1)(n)$ and $h_2 = (\mathrm{Opers}\, U_2)(n)$ holds $(\mathrm{Opers}(U_1, U_2))(n) =]\!] h_1, h_2 [\![$.

The following proposition is true

(2) If U_1 and U_2 are similar, then $\langle [\!:$ the carrier of U_1, the carrier of $U_2 :\!]$, $\mathrm{Opers}(U_1, U_2) \rangle$ is a strict universal algebra.

Let us consider U_1, U_2. Let us assume that U_1 and U_2 are similar. The functor $[\!: U_1, U_2 :\!]$ yielding a strict universal algebra is defined as follows:

(Def.5) $[\!: U_1, U_2 :\!] = \langle [\!:$ the carrier of U_1, the carrier of $U_2 :\!], \mathrm{Opers}(U_1, U_2) \rangle$.

Let A, B be non-empty set. The functor $\mathrm{Inv}(A, B)$ yielding a function from $[\!: A, B :\!]$ into $[\!: B, A :\!]$ is defined as follows:

(Def.6) For every element a of $[\!: A, B :\!]$ holds $(\mathrm{Inv}(A, B))(a) = \langle a_2, a_1 \rangle$.

One can prove the following propositions:

(3) For all non-empty set A, B holds rng $\mathrm{Inv}(A, B) = [\!: B, A :\!]$.

(4) For all non-empty set A, B holds $\mathrm{Inv}(A, B)$ is one-to-one.

(5) Suppose U_1 and U_2 are similar. Then Inv(the carrier of U_1, the carrier of U_2) is a function from the carrier of $[\!: U_1, U_2 :\!]$ into the carrier of $[\!: U_2, U_1 :\!]$.

(6) Suppose U_1 and U_2 are similar. Let o_1 be a operation of U_1, and let o_2 be a operation of U_2, and let o be a operation of $[\!: U_1, U_2 :\!]$, and let n be a natural number. Suppose $o_1 = (\mathrm{Opers}\, U_1)(n)$ and $o_2 = (\mathrm{Opers}\, U_2)(n)$ and $o = (\mathrm{Opers}[\!: U_1, U_2 :\!])(n)$ and $n \in$ dom Opers U_1. Then arity o = arity o_1 and arity o = arity o_2 and $o =]\!] o_1, o_2 [\![$.

(7) If U_1 and U_2 are similar, then $[\!: U_1, U_2 :\!]$ and U_1 are similar.

(8) Let U_1, U_2, U_3, U_4 be universal algebras. Suppose U_1 is a subalgebra of U_2 and U_3 is a subalgebra of U_4 and U_2 and U_4 are similar. Then $[\!: U_1, U_3 :\!]$ is a subalgebra of $[\!: U_2, U_4 :\!]$.

2. Trivial Algebra

Let k be a natural number. The functor TrivOp(k) yields a homogeneous quasi total non-empty partial function from $\{\emptyset\}^*$ to $\{\emptyset\}$ and is defined as follows:

(Def.7) dom TrivOp(k) $= \{k \longmapsto \emptyset\}$ and rng TrivOp(k) $= \{\emptyset\}$.

The following proposition is true

(9) arity TrivOp(k) $= k$.

Let f be a finite sequence of elements of $\mathbb{N}$. The functor TrivOps(f) yielding a finite sequence of elements of $\{\emptyset\}^* \dot{\rightarrow} \{\emptyset\}$ is defined as follows:

(Def.8) len TrivOps(f) = len f and for every n such that $n \in$ dom TrivOps(f) and for every m such that $m = f(n)$ holds (TrivOps(f))(n) = TrivOp(m).

We now state two propositions:

(10) For every finite sequence f of elements of $\mathbb{N}$ holds TrivOps(f) is homogeneous quasi total and non-empty.

(11) For every finite sequence f of elements of $\mathbb{N}$ such that $f \neq \varepsilon$ holds $\langle\{\emptyset\}$, TrivOps(f)$\rangle$ is a strict universal algebra.

Let D be a non empty set. Observe that there exists a finite sequence of elements of D which is non empty and there exists an element of D^* which is non empty.

Let f be a non empty finite sequence of elements of $\mathbb{N}$. The trivial algebra of f yielding a strict universal algebra is defined as follows:

(Def.9) The trivial algebra of $f = \langle\{\emptyset\}$, TrivOps(f)$\rangle$.

3. Product of Universal Algebras

A function is universal algebra yielding if:

(Def.10) For every x such that $x \in$ dom it holds it(x) is a universal algebra.

A function is 1-sorted yielding if:

(Def.11) For every x such that $x \in$ dom it holds it(x) is a 1-sorted structure.

One can check that there exists a function which is universal algebra yielding.

One can verify that every function which is universal algebra yielding is also 1-sorted yielding.

Let I be a set. Observe that there exists a many sorted set of I which is 1-sorted yielding.

A function is equal signature if:

(Def.12) For all x, y such that $x \in$ dom it and $y \in$ dom it and for all U_1, U_2 such that U_1 = it(x) and U_2 = it(y) holds signature U_1 = signature U_2.

Let J be a non-empty set. One can check that there exists a many sorted set of J which is equal signature and universal algebra yielding.

Let J be a non empty set, let A be a universal algebra yielding many sorted set of J, and let j be an element of J. Then $A(j)$ is a universal algebra.

Let J be a non-empty set and let A be a universal algebra yielding many sorted set of J. The functor support A yields a non-empty many sorted set of J and is defined as follows:

(Def.13) For every element j of J holds $(\text{support}\,A)(j) =$ the carrier of $A(j)$.

Let J be a non-empty set and let A be an equal signature universal algebra yielding many sorted set of J. The functor $\text{ComSign}(A)$ yields a finite sequence of elements of $\mathbb{N}$ and is defined as follows:

(Def.14) For every element j of J holds $\text{ComSign}(A) = \text{signature}\,A(j)$.

A function is function yielding if:

(Def.15) For every x such that $x \in \text{dom}\,\text{it}$ holds $\text{it}(x)$ is a function.

Let us note that there exists a function which is function yielding.

Let I be a set. Note that there exists a many sorted set of I which is function yielding.

Let I be a set. A many sorted function of I is a function yielding many sorted set of I.

Let J be a non-empty set, let B be a many sorted function of J, and let j be an element of J. Then $B(j)$ is a function.

Let J be a non-empty set, let B be a non-empty many sorted set of J, and let j be an element of J. Then $B(j)$ is a non-empty set.

Let J be a non-empty set and let B be a non-empty many sorted set of J. Then $\prod B$ is a non-empty set.

Let J be a non-empty set and let B be a non-empty many sorted set of J. A many sorted function of J is said to be a many sorted operation of B if:

(Def.16) For every element j of J holds $\text{it}(j)$ is a homogeneous quasi total non-empty partial function from $B(j)^*$ to $B(j)$.

Let J be a non-empty set, let B be a non-empty many sorted set of J, let O be a many sorted operation of B, and let j be an element of J. Then $O(j)$ is a homogeneous quasi total non-empty partial function from $B(j)^*$ to $B(j)$.

A function is equal arity if satisfies the condition (Def.17).

(Def.17) Let x, y be arbitrary. Suppose $x \in \text{dom}\,\text{it}$ and $y \in \text{dom}\,\text{it}$. Let f, g be functions. Suppose $\text{it}(x) = f$ and $\text{it}(y) = g$. Let n, m be natural numbers and let X, Y be non-empty set. Suppose $\text{dom}\,f = X^n$ and $\text{dom}\,g = Y^m$. Let o_1 be a homogeneous quasi total non-empty partial function from X^* to X and let o_2 be a homogeneous quasi total non-empty partial function from Y^* to Y. If $f = o_1$ and $g = o_2$, then $\text{arity}\,o_1 = \text{arity}\,o_2$.

Let J be a non-empty set and let B be a non-empty many sorted set of J. One can verify that there exists a many sorted operation of B which is equal arity.

The following proposition is true

(12) Let J be a non-empty set, and let B be a non-empty many sorted set of J, and let O be a many sorted operation of B. Then O is equal arity

if and only if for all elements i, j of J holds arity $O(i) =$ arity $O(j)$.

Let I be a set, let f be a many sorted function of I, and let x be a many sorted set of I. The functor $f \leftarrow\!\!\mathrm{P}\ x$ yields a many sorted set of I and is defined as follows:

(Def.18) For arbitrary i such that $i \in I$ and for every function g such that $g = f(i)$ holds $(f \leftarrow\!\!\mathrm{P}\ x)(i) = g(x(i))$.

Let J be a non-empty set, let B be a non-empty many sorted set of J, and let p be a finite sequence of elements of $\prod B$. Then uncurry p is a many sorted set of $[\!:\mathrm{dom}\,p, J:\!]$.

Let I, J be sets and let X be a many sorted set of $[\!:I, J:\!]$. Then $\curvearrowleft X$ is a many sorted set of $[\!:J, I:\!]$.

Let X be a set, let Y be a non-empty set, and let f be a many sorted set of $[\!:X, Y:\!]$. Then curry f is a many sorted set of X.

Let J be a non-empty set, let B be a non-empty many sorted set of J, and let O be an equal arity many sorted operation of B. The functor ComAr(O) yielding a natural number is defined as follows:

(Def.19) For every element j of J holds ComAr$(O) =$ arity $O(j)$.

Let I be a set and let A be a many sorted set of I. The functor ε_A yielding a many sorted set of I is defined as follows:

(Def.20) For arbitrary i such that $i \in I$ holds $\varepsilon_A(i) = \varepsilon_{A(i)}$.

Let J be a non-empty set, let B be a non-empty many sorted set of J, and let O be an equal arity many sorted operation of B. The functor $]\!|O|\![$ yielding a homogeneous quasi total non-empty partial function from $(\prod B)^*$ to $\prod B$ is defined by the conditions (Def.21).

(Def.21) (i) $\mathrm{dom}\,]\!|O|\![\, = (\prod B)^{\mathrm{ComAr}(O)}$, and

(ii) for every element p of $(\prod B)^*$ such that $p \in \mathrm{dom}\,]\!|O|\![$ holds if $\mathrm{dom}\,p = \emptyset$, then $]\!|O|\![(p) = O \leftarrow\!\!\mathrm{P}\ (\varepsilon_B)$ and if $\mathrm{dom}\,p \neq \emptyset$, then for every non-empty set Z and for every many sorted set w of $[\!:J, Z:\!]$ such that $Z = \mathrm{dom}\,p$ and $w = \curvearrowleft$uncurry p holds $]\!|O|\![(p) = O \leftarrow\!\!\mathrm{P}$ curry w.

Let J be a non-empty set, let A be an equal signature universal algebra yielding many sorted set of J, and let n be a natural number. Let us assume that $n \in$ Seg len ComSign(A). The functor ProdOp(A, n) yielding an equal arity many sorted operation of support A is defined by:

(Def.22) For every element j of J and for every operation o of $A(j)$ such that $(\mathrm{Opers}\,A(j))(n) = o$ holds $(\mathrm{ProdOp}(A,n))(j) = o$.

Let J be a non-empty set and let A be an equal signature universal algebra yielding many sorted set of J. The functor ProdOpSeq(A) yielding a finite sequence of elements of $(\prod \mathrm{support}\,A)^* \dot{\to} \prod \mathrm{support}\,A$ is defined as follows:

(Def.23) len ProdOpSeq$(A) =$ len ComSign(A) and for every n such that $n \in$ dom ProdOpSeq(A) holds $(\mathrm{ProdOpSeq}(A))(n) =]\!|\,\mathrm{ProdOp}(A,n)|\![$.

Let J be a non-empty set and let A be an equal signature universal algebra yielding many sorted set of J. The functor ProdUnivAlg(A) yields a strict universal algebra and is defined as follows:

(Def.24) $\text{ProdUnivAlg}(A) = \langle \prod \text{support}\, A, \text{ProdOpSeq}(A) \rangle.$

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Homomorphisms of Algebras. Quotient Universal Algebra

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Summary. The first part introduces homomorphisms of universal algebras and their basic properties. The second is concerned with the construction of a quotient universal algebra. The first isomorphism theorem is proved.

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The articles [9], [10], [11], [4], [5], [1], [8], [3], [6], [7], and [2] provide the terminology and notation for this paper.

1. Homomorphisms of Algebras

For simplicity we adopt the following convention: U_1, U_2, U_3 will denote universal algebras, n will denote a natural number, o_1 will denote a operation of U_1, o_2 will denote a operation of U_2, and x, y will be arbitrary.

Let D_1, D_2 be non empty set, let p be a finite sequence of elements of D_1, and let f be a function from D_1 into D_2. Then $f \cdot p$ is a finite sequence of elements of D_2.

The following propositions are true:

(1) Let D_1, D_2 be non empty set, and let p be a finite sequence of elements of D_1, and let f be a function from D_1 into D_2. Then $\mathrm{dom}(f \cdot p) = \mathrm{dom}\, p$ and $\mathrm{len}(f \cdot p) = \mathrm{len}\, p$ and for every n such that $n \in \mathrm{dom}(f \cdot p)$ holds $(f \cdot p)(n) = f(p(n))$.

(2) For every non empty subset B of U_1 such that $B =$ the carrier of U_1 holds $\mathrm{Opers}(U_1, B) = \mathrm{Opers}\, U_1$.

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Let U_1 be a universal algebra. A finite sequence of elements of U_1 is a finite sequence of elements of the carrier of U_1. Let U_2 be a universal algebra. A function from U_1 into U_2 is a function from the carrier of U_1 into the carrier of U_2.

In the sequel a, a_1, a_2 denote finite sequences of elements of U_1 and f denotes a function from U_1 into U_2.

One can prove the following three propositions:

(3) $f \cdot \varepsilon_{(\text{the carrier of } U_1)} = \varepsilon_{(\text{the carrier of } U_2)}$.

(4) $\mathrm{id}_{(\text{the carrier of } U_1)} \cdot a = a$.

(5) Let h_1 be a function from U_1 into U_2, and let h_2 be a function from U_2 into U_3, and let a be a finite sequence of elements of U_1. Then $h_2 \cdot (h_1 \cdot a) = (h_2 \cdot h_1) \cdot a$.

Let us consider U_1, U_2, f. We say that f is a homomorphism of U_1 into U_2 if and only if the conditions (Def.1) are satisfied.

(Def.1) (i) U_1 and U_2 are similar, and

(ii) for every n such that $n \in \mathrm{dom}\,\mathrm{Opers}\,U_1$ and for all o_1, o_2 such that $o_1 = (\mathrm{Opers}\,U_1)(n)$ and $o_2 = (\mathrm{Opers}\,U_2)(n)$ and for every finite sequence x of elements of U_1 such that $x \in \mathrm{dom}\,o_1$ holds $f(o_1(x)) = o_2(f \cdot x)$.

Let us consider U_1, U_2, f. We say that f is a monomorphism of U_1 into U_2 if and only if:

(Def.2) f is a homomorphism of U_1 into U_2 and one-to-one.

We say that f is an epimorphism of U_1 onto U_2 if and only if:

(Def.3) f is a homomorphism of U_1 into U_2 and $\mathrm{rng}\, f =$ the carrier of U_2.

Let us consider U_1, U_2, f. We say that f is an isomorphism of U_1 and U_2 if and only if:

(Def.4) f is a monomorphism of U_1 into U_2 and an epimorphism of U_1 onto U_2.

Let us consider U_1, U_2. We say that U_1 and U_2 are isomorphic if and only if:

(Def.5) There exists f which is an isomorphism of U_1 and U_2.

One can prove the following propositions:

(6) $\mathrm{id}_{(\text{the carrier of } U_1)}$ is a homomorphism of U_1 into U_1.

(7) Let h_1 be a function from U_1 into U_2 and let h_2 be a function from U_2 into U_3. Suppose h_1 is a homomorphism of U_1 into U_2 and h_2 is a homomorphism of U_2 into U_3. Then $h_2 \cdot h_1$ is a homomorphism of U_1 into U_3.

(8) f is an isomorphism of U_1 and U_2 if and only if f is a homomorphism of U_1 into U_2 and $\mathrm{rng}\, f =$ the carrier of U_2 and f is one-to-one.

(9) If f is an isomorphism of U_1 and U_2, then $\mathrm{dom}\, f =$ the carrier of U_1 and $\mathrm{rng}\, f =$ the carrier of U_2.

(10) Let h be a function from U_1 into U_2 and let h_1 be a function from U_2 into U_1. Suppose h is an isomorphism of U_1 and U_2 and $h_1 = h^{-1}$. Then h_1 is a homomorphism of U_2 into U_1.

(11) Let h be a function from U_1 into U_2 and let h_1 be a function from U_2 into U_1. Suppose h is an isomorphism of U_1 and U_2 and $h_1 = h^{-1}$. Then h_1 is an isomorphism of U_2 and U_1.

(12) Let h be a function from U_1 into U_2 and let h_1 be a function from U_2 into U_3. Suppose h is an isomorphism of U_1 and U_2 and h_1 is an isomorphism of U_2 and U_3. Then $h_1 \cdot h$ is an isomorphism of U_1 and U_3.

(13) U_1 and U_1 are isomorphic.

(14) If U_1 and U_2 are isomorphic, then U_2 and U_1 are isomorphic.

(15) If U_1 and U_2 are isomorphic and U_2 and U_3 are isomorphic, then U_1 and U_3 are isomorphic.

Let us consider U_1, U_2, f. Let us assume that f is a homomorphism of U_1 into U_2. The functor $\mathrm{Im}\, f$ yielding a strict subalgebra of U_2 is defined as follows:

(Def.6) The carrier of $\mathrm{Im}\, f = f^\circ$ (the carrier of U_1).

Next we state two propositions:

(16) For every function h from U_1 into U_2 such that h is a homomorphism of U_1 into U_2 holds $\mathrm{rng}\, h =$ the carrier of $\mathrm{Im}\, h$.

(17) Let U_2 be a strict universal algebra and let f be a function from U_1 into U_2. Suppose f is a homomorphism of U_1 into U_2. Then f is an epimorphism of U_1 onto U_2 if and only if $\mathrm{Im}\, f = U_2$.

2. Quotient Universal Algebra

Let us consider U_1. A binary relation on U_1 is a binary relation on the carrier of U_1. An equivalence relation of U_1 is an equivalence relation of the carrier of U_1.

Let D be a non empty set and let R be a binary relation on D. The functor $R^\#$ yielding a binary relation on D^* is defined by the condition (Def.7).

(Def.7) Let x, y be finite sequences of elements of D. Then $\langle x, y \rangle \in R^\#$ if and only if the following conditions are satisfied:

(i) $\mathrm{len}\, x = \mathrm{len}\, y$, and

(ii) for every n such that $n \in \mathrm{dom}\, x$ holds $\langle x(n), y(n) \rangle \in R$.

The following proposition is true

(18) For every non empty set D holds $(\triangle_D)^\# = \triangle_{D^*}$.

Let us consider U_1. An equivalence relation of U_1 is said to be a congruence of U_1 if it satisfies the condition (Def.8).

(Def.8) Given n, o_1. Suppose $n \in \mathrm{dom\, Opers}\, U_1$ and $o_1 = (\mathrm{Opers}\, U_1)(n)$. Let x, y be finite sequences of elements of U_1. If $x \in \mathrm{dom}\, o_1$ and $y \in \mathrm{dom}\, o_1$ and $\langle x, y \rangle \in \mathrm{it}^\#$, then $\langle o_1(x), o_1(y) \rangle \in \mathrm{it}$.

Let D be a non empty set and let R be an equivalence relation of D. Then $\mathrm{Classes}\, R$ is a non empty family of subsets of D.

Let D be a non empty set, let R be an equivalence relation of D, let y be a finite sequence of elements of Classes R, and let x be a finite sequence of elements of D. We say that x is a finite sequence of representatives of y if and only if:

(Def.9) $\operatorname{len} x = \operatorname{len} y$ and for every n such that $n \in \operatorname{dom} x$ holds $[x(n)]_R = y(n)$.

We now state the proposition

(19) Let D be a non empty set, and let R be an equivalence relation of D, and let y be a finite sequence of elements of Classes R. Then there exists finite sequence of elements of D which is a finite sequence of representatives of y.

Let U_1 be a universal algebra, let E be a congruence of U_1, and let o be a operation of U_1. The functor $o_{/E}$ yields a homogeneous quasi total non-empty partial function from (Classes E)* to Classes E and is defined by the conditions (Def.10).

(Def.10) (i) $\operatorname{dom}(o_{/E}) = (\text{Classes } E)^{\text{arity}\, o}$, and

(ii) for every finite sequence y of elements of Classes E such that $y \in \operatorname{dom}(o_{/E})$ and for every finite sequence x of elements of the carrier of U_1 such that x is a finite sequence of representatives of y holds $o_{/E}(y) = [o(x)]_E$.

Let us consider U_1, E. The functor $\operatorname{Opers}(U_1)_{/E}$ yields a finite sequence of elements of $(\text{Classes } E)^* \dot{\to} \text{Classes } E$ and is defined as follows:

(Def.11) $\operatorname{len}(\operatorname{Opers}((U_1))_{/E}) = \operatorname{len} \operatorname{Opers} U_1$ and for every n such that $n \in \operatorname{dom}(\operatorname{Opers}((U_1))_{/E})$ and for every o_1 such that $(\operatorname{Opers} U_1)(n) = o_1$ holds $\operatorname{Opers}((U_1))_{/E}(n) = (o_1)_{/E}$.

Next we state the proposition

(20) For all U_1, E holds $\langle \text{Classes } E, \operatorname{Opers}((U_1))_{/E} \rangle$ is a strict universal algebra.

Let us consider U_1, E. The functor $U_{1/E}$ yielding a strict universal algebra is defined by:

(Def.12) $(U_1)_{/E} = \langle \text{Classes } E, \operatorname{Opers}((U_1))_{/E} \rangle$.

Let us consider U_1, E. The natural homomorphism of U_1 w.r.t. E yielding a function from U_1 into $(U_1)_{/E}$ is defined as follows:

(Def.13) For every element u of the carrier of U_1 holds (the natural homomorphism of U_1 w.r.t. E)$(u) = [u]_E$.

One can prove the following two propositions:

(21) For all U_1, E holds the natural homomorphism of U_1 w.r.t. E is a homomorphism of U_1 into $(U_1)_{/E}$.

(22) For all U_1, E holds the natural homomorphism of U_1 w.r.t. E is an epimorphism of U_1 onto $(U_1)_{/E}$.

Let us consider U_1, U_2 and let f be a function from U_1 into U_2. Let us assume that f is a homomorphism of U_1 into U_2. The functor $\operatorname{Cng}(f)$ yielding a congruence of U_1 is defined by:

(Def.14) For all elements a, b of the carrier of U_1 holds $\langle a, b \rangle \in \mathrm{Cng}(f)$ iff $f(a) = f(b)$.

Let U_1, U_2 be universal algebras and let f be a function from U_1 into U_2. Let us assume that f is a homomorphism of U_1 into U_2. The functor $\overline{f}$ yielding a function from $(U_1)_{/\,\mathrm{Cng}(f)}$ into U_2 is defined by:

(Def.15) For every element a of the carrier of U_1 holds $(\overline{f})([a]_{\mathrm{Cng}(f)}) = f(a)$.

We now state three propositions:

(23) Suppose f is a homomorphism of U_1 into U_2. Then $\overline{f}$ is a homomorphism of $(U_1)_{/\,\mathrm{Cng}(f)}$ into U_2 and $\overline{f}$ is a monomorphism of $(U_1)_{/\,\mathrm{Cng}(f)}$ into U_2.

(24) If f is an epimorphism of U_1 onto U_2, then $\overline{f}$ is an isomorphism of $(U_1)_{/\,\mathrm{Cng}(f)}$ and U_2.

(25) If f is an epimorphism of U_1 onto U_2, then $(U_1)_{/\,\mathrm{Cng}(f)}$ and U_2 are isomorphic.

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Free Universal Algebra Construction

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Summary. A construction of the free universal algebra with fixed signature and a given set of generators.

MML Identifier: FREEALG.

The articles [17], [19], [20], [9], [13], [10], [11], [5], [16], [8], [18], [1], [3], [4], [2], [15], [7], [12], [6], and [14] provide the terminology and notation for this paper.

1. Preliminaries

In the sequel x is arbitrary and n denotes a natural number.

Let D be a non empty set and let X be a set. Then $D \cup X$ is a non empty set.

A set is missing $\mathbb{N}$ if:

(Def.1) It $\cap \mathbb{N} = \emptyset$.

One can check that there exists a set which is non empty and missing $\mathbb{N}$.

A finite sequence has zero if:

(Def.2) $0 \in \operatorname{rng}$ it.

Let us observe that there exists a finite sequence of elements of $\mathbb{N}$ which is non empty and has zero and there exists a finite sequence of elements of $\mathbb{N}$ which is non empty and without zero.

Let f be a non empty finite sequence. Then $\operatorname{dom} f$ is a non empty set.

Let X be a set, let D be a non empty set, let f be a partial function from X to D, and let x be arbitrary. Let us assume that $x \in \operatorname{dom} f$. The functor $\pi_x f$ yields an element of D and is defined as follows:

(Def.3) $\pi_x f = f(x)$.

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2. Free Universal Algebra - General Notions

Let U_1 be a universal algebra and let n be a natural number. Let us assume that $n \in \mathrm{dom\,Opers}\,U_1$. The functor $\mathrm{oper}(n, U_1)$ yielding a operation of U_1 is defined as follows:

(Def.4) $\mathrm{oper}(n, U_1) = (\mathrm{Opers}\,U_1)(n)$.

Let U_0 be a universal algebra. A subset of U_0 is called a generator set of U_0 if:

(Def.5) The carrier of $\mathrm{Gen}^{\mathrm{UA}}(\mathrm{it})$ = the carrier of U_0.

Let U_0 be a universal algebra. A generator set of U_0 is free if satisfies the condition (Def.6).

(Def.6) Let U_1 be a universal algebra. Suppose U_0 and U_1 are similar. Let f be a function from it into the carrier of U_1. Then there exists a function h from U_0 into U_1 such that h is a homomorphism of U_0 into U_1 and $h \restriction \mathrm{it} = f$.

A universal algebra is free if:

(Def.7) There exists generator set of it which is free.

Let us observe that there exists a universal algebra which is free and strict.

Let U_0 be a free universal algebra. Observe that there exists a generator set of U_0 which is free.

One can prove the following proposition

(1) Let U_0 be a strict universal algebra and let A be a subset of U_0. Then A is a generator set of U_0 if and only if $\mathrm{Gen}^{\mathrm{UA}}(A) = U_0$.

3. Construction of Decorated Tree Structure for Free Universal Algebra

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let X be a set. The functor $\mathrm{REL}(f, X)$ yielding a relation between $\mathrm{dom}\,f \cup X$ and $(\mathrm{dom}\,f \cup X)^*$ is defined by:

(Def.8) For every element a of $\mathrm{dom}\,f \cup X$ and for every element b of $(\mathrm{dom}\,f \cup X)^*$ holds $\langle a, b\rangle \in \mathrm{REL}(f, X)$ iff $a \in \mathrm{dom}\,f$ and $f(a) = \mathrm{len}\,b$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let X be a set. The functor $\mathrm{DTConUA}(f, X)$ yields a strict tree construction structure and is defined as follows:

(Def.9) $\mathrm{DTConUA}(f, X) = \langle \mathrm{dom}\,f \cup X, \mathrm{REL}(f, X)\rangle$.

Next we state two propositions:

(2) Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let X be a set. Then the terminals of $\mathrm{DTConUA}(f, X) \subseteq X$ and the nonterminals of $\mathrm{DTConUA}(f, X) = \mathrm{dom}\,f$.

(3) Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let X be a missing $\mathbb{N}$ set. Then the terminals of $\mathrm{DTConUA}(f, X) = X$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let X be a set. Then $\mathrm{DTConUA}(f, X)$ is a strict tree construction structure with nonterminals.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let X be a set. Then $\mathrm{DTConUA}(f, X)$ is a strict tree construction structure with nonterminals and useful nonterminals.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a missing $\mathbb{N}$ non empty set. Then $\mathrm{DTConUA}(f, D)$ is a strict tree construction structure with terminals, nonterminals, and useful nonterminals.

Let f be a non empty finite sequence of elements of $\mathbb{N}$, let X be a set, and let n be a natural number. Let us assume that $n \in \mathrm{dom}\, f$. The functor $\mathrm{Sym}(n, f, X)$ yielding a symbol of $\mathrm{DTConUA}(f, X)$ is defined by:

(Def.10) $\mathrm{Sym}(n, f, X) = n$.

4. Construction of Free Universal Algebra for Non-empty Set of Generators and Given Signature

Let f be a non empty finite sequence of elements of $\mathbb{N}$, let D be a missing $\mathbb{N}$ non empty set, and let n be a natural number. Let us assume that $n \in \mathrm{dom}\, f$. The functor $\mathrm{FreeOpNSG}(n, f, D)$ yields a homogeneous quasi total non empty partial function from $\mathrm{TS}(\mathrm{DTConUA}(f, D))^*$ to $\mathrm{TS}(\mathrm{DTConUA}(f, D))$ and is defined by the conditions (Def.11).

(Def.11) (i) $\mathrm{dom}\, \mathrm{FreeOpNSG}(n, f, D) = \mathrm{TS}(\mathrm{DTConUA}(f, D))^{\pi_n f}$, and

(ii) for every finite sequence p of elements of $\mathrm{TS}(\mathrm{DTConUA}(f, D))$ such that $p \in \mathrm{dom}\, \mathrm{FreeOpNSG}(n, f, D)$ holds $(\mathrm{FreeOpNSG}(n, f, D))(p) = (\mathrm{Sym}(n, f, D))\text{-tree}(p)$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a missing $\mathbb{N}$ non empty set. The functor $\mathrm{FreeOpSeqNSG}(f, D)$ yielding a finite sequence of elements of $\mathrm{TS}(\mathrm{DTConUA}(f, D))^* \dot{\rightarrow} \mathrm{TS}(\mathrm{DTConUA}(f, D))$ is defined as follows:

(Def.12) $\mathrm{len}\, \mathrm{FreeOpSeqNSG}(f, D) = \mathrm{len}\, f$ and for every n such that $n \in \mathrm{dom}\, \mathrm{FreeOpSeqNSG}(f, D)$ holds $(\mathrm{FreeOpSeqNSG}(f, D))(n) = \mathrm{FreeOpNSG}(n, f, D)$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a missing $\mathbb{N}$ non empty set. The functor $\mathrm{FreeUnivAlgNSG}(f, D)$ yields a strict universal algebra and is defined as follows:

(Def.13) $\mathrm{FreeUnivAlgNSG}(f, D) = \langle \mathrm{TS}(\mathrm{DTConUA}(f, D)), \mathrm{FreeOpSeqNSG}(f, D) \rangle$.

One can prove the following proposition

(4) For every non empty finite sequence f of elements of $\mathbb{N}$ and for every missing $\mathbb{N}$ non empty set D holds $\mathrm{signature}\, \mathrm{FreeUnivAlgNSG}(f, D) = f$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a non empty missing $\mathbb{N}$ set. The functor FreeGenSetNSG(f, D) yielding a subset of FreeUnivAlgNSG(f, D) is defined by:

(Def.14) FreeGenSetNSG(f, D) = {the root tree of s: s ranges over symbols of DTConUA(f, D), $s \in$ the terminals of DTConUA(f, D)}.

One can prove the following proposition

(5) Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a non empty missing $\mathbb{N}$ set. Then FreeGenSetNSG(f, D) is non empty.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a non empty missing $\mathbb{N}$ set. Then FreeGenSetNSG(f, D) is a generator set of FreeUnivAlgNSG(f, D).

Let f be a non empty finite sequence of elements of $\mathbb{N}$, let D be a non empty missing $\mathbb{N}$ set, let C be a non empty set, let s be a symbol of DTConUA(f, D), and let F be a function from FreeGenSetNSG(f, D) into C. Let us assume that $s \in$ the terminals of DTConUA(f, D). The functor $\pi_s F$ yielding an element of C is defined as follows:

(Def.15) $\pi_s F = F$(the root tree of s).

Let f be a non empty finite sequence of elements of $\mathbb{N}$, let D be a non empty missing $\mathbb{N}$ set, and let s be a symbol of DTConUA(f, D). Let us assume that there exists a finite sequence p such that $s \Rightarrow p$. The functor ${}^{@}s$ yielding a natural number is defined by:

(Def.16) ${}^{@}s = s$.

Next we state the proposition

(6) For every non empty finite sequence f of elements of $\mathbb{N}$ and for every non empty missing $\mathbb{N}$ set D holds FreeGenSetNSG(f, D) is free.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a non empty missing $\mathbb{N}$ set. Then FreeUnivAlgNSG(f, D) is a strict free universal algebra.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a non empty missing $\mathbb{N}$ set. Then FreeGenSetNSG(f, D) is a free generator set of FreeUnivAlgNSG(f, D).

5. Construction of Free Universal Algebra and Set of Generators

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero, let D be a missing $\mathbb{N}$ set, and let n be a natural number. Let us assume that $n \in \operatorname{dom} f$. The functor FreeOpZAO($n, f, D$) yields a homogeneous quasi total non empty partial function from TS(DTConUA(f, D))* to TS(DTConUA(f, D)) and is defined by the conditions (Def.17).

(Def.17) (i) dom FreeOpZAO(n, f, D) = TS(DTConUA(f, D))$^{\pi_n f}$, and

(ii) for every finite sequence p of elements of TS(DTConUA(f, D)) such that $p \in$ dom FreeOpZAO(n, f, D) holds (FreeOpZAO(n, f, D))(p) = (Sym(n, f, D))-tree(p).

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. The functor FreeOpSeqZAO(f, D) yields a finite sequence of elements of TS(DTConUA(f, D))$^* \dot{\rightarrow}$ TS(DTConUA(f, D)) and is defined by:

(Def.18) len FreeOpSeqZAO(f, D) = len f and for every n such that $n \in$ dom FreeOpSeqZAO(f, D) holds (FreeOpSeqZAO(f, D))(n) = FreeOpZAO(n, f, D).

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. The functor FreeUnivAlgZAO(f, D) yielding a strict universal algebra is defined by:

(Def.19) FreeUnivAlgZAO(f, D) = $\langle$TS(DTConUA(f, D)), FreeOpSeqZAO(f, D)$\rangle$

We now state three propositions:

(7) For every non empty finite sequence f of elements of $\mathbb{N}$ with zero and for every missing $\mathbb{N}$ set D holds signature FreeUnivAlgZAO(f, D) = f.

(8) Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. Then FreeUnivAlgZAO(f, D) has constants.

(9) For every non empty finite sequence f of elements of $\mathbb{N}$ with zero and for every missing $\mathbb{N}$ set D holds Constants(FreeUnivAlgZAO(f, D)) $\neq \emptyset$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. The functor FreeGenSetZAO(f, D) yielding a subset of FreeUnivAlgZAO(f, D) is defined as follows:

(Def.20) FreeGenSetZAO(f, D) = {the root tree of s: s ranges over symbols of DTConUA(f, D), $s \in$ the terminals of DTConUA(f, D)}.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. Then FreeGenSetZAO(f, D) is a generator set of FreeUnivAlgZAO(f, D).

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero, let D be a missing $\mathbb{N}$ set, let C be a non empty set, let s be a symbol of DTConUA(f, D), and let F be a function from FreeGenSetZAO(f, D) into C. Let us assume that $s \in$ the terminals of DTConUA(f, D). The functor $\pi_s F$ yields an element of C and is defined by:

(Def.21) $\pi_s F = F$(the root tree of s).

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero, let D be a missing $\mathbb{N}$ set, and let s be a symbol of DTConUA(f, D). Let us assume that there exists a finite sequence p such that $s \Rightarrow p$. The functor $^{@}s$ yields a natural number and is defined by:

(Def.22) $^{@}s = s$.

The following proposition is true

(10) For every non empty finite sequence f of elements of $\mathbb{N}$ with zero and for every missing $\mathbb{N}$ set D holds FreeGenSetZAO(f, D) is free.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. Then FreeUnivAlgZAO(f, D) is a strict free universal algebra.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. Then FreeGenSetZAO(f, D) is a free generator set of FreeUnivAlgZAO(f, D).

One can verify that there exists a universal algebra which is strict and free and has constants.

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Complex Sequences

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Summary. Definitions of complex sequence and operations on sequences (multiplication of sequences and multiplication by a complex number, addition, subtraction, division and absolute value of sequence) are given. We followed [3].

MML Identifier: COMSEQ_1.

The terminology and notation used here are introduced in the following articles: [5], [1], [2], [4], and [3].

For simplicity we follow a convention: f will denote a function, n will denote a natural number, r, p will denote elements of $\mathbb{C}$, and x will be arbitrary.

A complex sequence is a function from $\mathbb{N}$ into $\mathbb{C}$.

In the sequel s_1, s_2, s_3, s_4, s'_1, s'_2 denote complex sequences.

One can prove the following propositions:

(1) f is a complex sequence iff $\operatorname{dom} f = \mathbb{N}$ and for every x such that $x \in \mathbb{N}$ holds $f(x)$ is an element of $\mathbb{C}$.

(2) f is a complex sequence iff $\operatorname{dom} f = \mathbb{N}$ and for every n holds $f(n)$ is an element of $\mathbb{C}$.

Let us consider s_1, n. Then $s_1(n)$ is an element of $\mathbb{C}$.

The scheme *ExComplexSeq* deals with a unary functor $\mathcal{F}$ yielding an element of $\mathbb{C}$, and states that:

There exists s_1 such that for every n holds $s_1(n) = \mathcal{F}(n)$

for all values of the parameter.

A complex sequence is non-zero if:

(Def.1) $\operatorname{rng} \mathrm{it} \subseteq \mathbb{C} \setminus \{0_{\mathbb{C}}\}$.

One can prove the following proposition

(3) s_1 is non-zero iff for every x such that $x \in \mathbb{N}$ holds $s_1(x) \neq 0_{\mathbb{C}}$.

Let us mention that there exists a complex sequence which is non-zero.

Next we state four propositions:

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(4) s_1 is non-zero iff for every n holds $s_1(n) \neq 0_{\mathbb{C}}$.

(5) For all s_1, s_2 such that for every x such that $x \in \mathbb{N}$ holds $s_1(x) = s_2(x)$ holds $s_1 = s_2$.

(6) For all s_1, s_2 such that for every n holds $s_1(n) = s_2(n)$ holds $s_1 = s_2$.

(7) For every r there exists s_1 such that $\operatorname{rng} s_1 = \{r\}$.

Let us consider s_2, s_3. The functor $s_2 + s_3$ yielding a complex sequence is defined as follows:

(Def.2) For every n holds $(s_2 + s_3)(n) = s_2(n) + s_3(n)$.

The functor $s_2\, s_3$ yielding a complex sequence is defined by:

(Def.3) For every n holds $(s_2\, s_3)(n) = s_2(n) \cdot s_3(n)$.

Let us consider r, s_1. The functor $r\, s_1$ yielding a complex sequence is defined as follows:

(Def.4) For every n holds $(r\, s_1)(n) = r \cdot s_1(n)$.

Let us consider s_1. The functor $-s_1$ yielding a complex sequence is defined as follows:

(Def.5) For every n holds $(-s_1)(n) = -s_1(n)$.

Let us consider s_2, s_3. The functor $s_2 - s_3$ yields a complex sequence and is defined as follows:

(Def.6) $s_2 - s_3 = s_2 + -s_3$.

Let us consider s_1. The functor ${s_1}^{-1}$ yields a complex sequence and is defined as follows:

(Def.7) For every n holds ${s_1}^{-1}(n) = s_1(n)^{-1}$.

Let us consider s_2, s_1. The functor $\frac{s_2}{s_1}$ yielding a complex sequence is defined as follows:

(Def.8) $\frac{s_2}{s_1} = s_2\, {s_1}^{-1}$.

Let us consider s_1. The functor $|s_1|$ yields a sequence of real numbers and is defined by:

(Def.9) For every n holds $|s_1|(n) = |s_1(n)|$.

The following propositions are true:

(8) $s_2 + s_3 = s_3 + s_2$.

(9) $(s_2 + s_3) + s_4 = s_2 + (s_3 + s_4)$.

(10) $s_2\, s_3 = s_3\, s_2$.

(11) $(s_2\, s_3)\, s_4 = s_2\, (s_3\, s_4)$.

(12) $(s_2 + s_3)\, s_4 = s_2\, s_4 + s_3\, s_4$.

(13) $s_4\, (s_2 + s_3) = s_4\, s_2 + s_4\, s_3$.

(14) $-s_1 = (-1_{\mathbb{C}})\, s_1$.

(15) $r\, (s_2\, s_3) = (r\, s_2)\, s_3$.

(16) $r\, (s_2\, s_3) = s_2\, (r\, s_3)$.

(17) $(s_2 - s_3)\, s_4 = s_2\, s_4 - s_3\, s_4$.

(18) $s_4\, s_2 - s_4\, s_3 = s_4\, (s_2 - s_3)$.

(19) $r(s_2 + s_3) = r\,s_2 + r\,s_3$.
(20) $(r \cdot p)\,s_1 = r\,(p\,s_1)$.
(21) $r(s_2 - s_3) = r\,s_2 - r\,s_3$.
(22) If s_1 is non-zero, then $r\,\frac{s_2}{s_1} = \frac{r\,s_2}{s_1}$.
(23) $s_2 - (s_3 + s_4) = s_2 - s_3 - s_4$.
(24) $1_{\mathbb{C}}\,s_1 = s_1$.
(25) $--s_1 = s_1$.
(26) $s_2 - -s_3 = s_2 + s_3$.
(27) $s_2 - (s_3 - s_4) = (s_2 - s_3) + s_4$.
(28) $s_2 + (s_3 - s_4) = (s_2 + s_3) - s_4$.
(29) $(-s_2)\,s_3 = -s_2\,s_3$ and $s_2\,-s_3 = -s_2\,s_3$.
(30) If s_1 is non-zero, then ${s_1}^{-1}$ is non-zero.
(31) If s_1 is non-zero, then $({s_1}^{-1})^{-1} = s_1$.
(32) s_1 is non-zero and s_2 is non-zero iff $s_1\,s_2$ is non-zero.
(33) If s_1 is non-zero and s_2 is non-zero, then ${s_1}^{-1}\,{s_2}^{-1} = (s_1\,s_2)^{-1}$.
(34) If s_1 is non-zero, then $\frac{s_2}{s_1}\,s_1 = s_2$.
(35) If s_1 is non-zero and s_2 is non-zero, then $\frac{s'_1}{s_1}\,\frac{s'_2}{s_2} = \frac{s'_1\,s'_2}{s_1\,s_2}$.
(36) If s_1 is non-zero and s_2 is non-zero, then $\frac{s_1}{s_2}$ is non-zero.
(37) If s_1 is non-zero and s_2 is non-zero, then $(\frac{s_1}{s_2})^{-1} = \frac{s_2}{s_1}$.
(38) If s_1 is non-zero, then $s_3\,\frac{s_2}{s_1} = \frac{s_3\,s_2}{s_1}$.
(39) If s_1 is non-zero and s_2 is non-zero, then $\frac{\frac{s_3}{\frac{s_1}{s_2}}}{} = \frac{s_3\,s_2}{s_1}$.
(40) If s_1 is non-zero and s_2 is non-zero, then $\frac{s_3}{s_1} = \frac{s_3\,s_2}{s_1\,s_2}$.
(41) If $r \neq 0_{\mathbb{C}}$ and s_1 is non-zero, then $r\,s_1$ is non-zero.
(42) If s_1 is non-zero, then $-s_1$ is non-zero.
(43) If $r \neq 0_{\mathbb{C}}$ and s_1 is non-zero, then $(r\,s_1)^{-1} = r^{-1}\,{s_1}^{-1}$.
(44) If s_1 is non-zero, then $(-s_1)^{-1} = (-1_{\mathbb{C}})\,{s_1}^{-1}$.
(45) If s_1 is non-zero, then $-\frac{s_2}{s_1} = \frac{-s_2}{s_1}$ and $\frac{s_2}{-s_1} = -\frac{s_2}{s_1}$.
(46) If s_1 is non-zero, then $\frac{s_2}{s_1} + \frac{s'_2}{s_1} = \frac{s_2+s'_2}{s_1}$ and $\frac{s_2}{s_1} - \frac{s'_2}{s_1} = \frac{s_2-s'_2}{s_1}$.
(47) If s_1 is non-zero and s'_1 is non-zero, then $\frac{s_2}{s_1} + \frac{s'_2}{s'_1} = \frac{s_2\,s'_1+s'_2\,s_1}{s_1\,s'_1}$ and $\frac{s_2}{s_1} - \frac{s'_2}{s'_1} = \frac{s_2\,s'_1-s'_2\,s_1}{s_1\,s'_1}$.
(48) If s_1 is non-zero and s'_1 is non-zero and s_2 is non-zero, then $\frac{\frac{s'_2}{s_1}}{\frac{s'_1}{s_2}} = \frac{s'_2\,s_2}{s_1\,s'_1}$.
(49) $|s_1\,s'_1| = |s_1|\,|s'_1|$.
(50) If s_1 is non-zero, then $|s_1|$ is non-zero.
(51) If s_1 is non-zero, then $|s_1|^{-1} = |{s_1}^{-1}|$.
(52) If s_1 is non-zero, then $|\frac{s'_1}{s_1}| = \frac{|s'_1|}{|s_1|}$.

(53) $|r\, s_1| = |r|\, |s_1|$.

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Maximal Discrete Subspaces of Almost Discrete Topological Spaces

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Summary. Let X be a topological space and let D be a subset of X. D is said to be *discrete* provided for every subset A of X such that $A \subseteq D$ there is an open subset G of X such that $A = D \cap G$ (comp. e.g., [7]). A discrete subset M of X is said to be *maximal discrete* provided for every discrete subset D of X if $M \subseteq D$ then $M = D$. A subspace of X is *discrete* (*maximal discrete*) iff its carrier is discrete (maximal discrete) in X.

Our purpose is to list a number of properties of discrete and maximal discrete sets in Mizar formalism. In particular, we show here that *if D is dense and discrete then D is maximal discrete*; moreover, *if D is open and maximal discrete then D is dense.* We discuss also the problem of the existence of maximal discrete subsets in a topological space.

To present the main results we first recall a definition of a class of topological spaces considered herein. A topological space X is called *almost discrete* if every open subset of X is closed; equivalently, if every closed subset of X is open. Such spaces were investigated in Mizar formalism in [4] and [5]. We show here that *every almost discrete space contains a maximal discrete subspace and every such subspace is a retract of the enveloping space.* Moreover, *if X_0 is a maximal discrete subspace of an almost discrete space X and $r : X \to X_0$ is a continuous retraction, then $r^{-1}(x) = \overline{\{x\}}$ for every point x of X belonging to X_0.* This fact is a specialization, in the case of almost discrete spaces, of the theorem of M.H. Stone that every topological space can be made into a T_0-space by suitable identification of points (see [9]).

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The terminology and notation used in this paper are introduced in the following papers: [13], [14], [10], [2], [3], [12], [1], [8], [15], [11], [4], and [6].

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1. Proper Subsets of 1-sorted Structures

A non empty set is trivial if:

(Def.1) There exists an element s of it such that it $= \{s\}$.

Let us note that there exists a non empty set which is trivial and there exists a non empty set which is non trivial.

Next we state four propositions:

(1) For every non empty set A and for every trivial non empty set B such that $A \subseteq B$ holds $A = B$.

(2) For every trivial non empty set A and for every set B such that $A \cap B$ is non empty holds $A \subseteq B$.

(3) For every 1-sorted structure Y holds Y is trivial iff the carrier of Y is trivial.

(4) Let Y_0, Y_1 be 1-sorted structures. Suppose the carrier of Y_0 = the carrier of Y_1. If Y_0 is trivial, then Y_1 is trivial.

Let S be a set. An element of S is proper if:

(Def.2) It $\neq \bigcup S$.

Let S be a set. Observe that there exists a subset of S which is non proper.

Next we state the proposition

(5) For every set S and for every subset A of S holds A is proper iff $A \neq S$.

Let S be a non empty set. Observe that every subset of S which is non proper is also non empty and every subset of S which is empty is also proper.

Let S be a trivial non empty set. Observe that every subset of S which is proper is also empty and every subset of S which is non empty is also non proper.

Let S be a non empty set. One can check that there exists a subset of S which is proper and there exists a subset of S which is non proper.

Let S be a non empty set and let y be an element of S. Then $\{y\}$ is a non empty subset of S.

Let S be a non empty set. Observe that there exists a non empty subset of S which is trivial.

Let S be a non empty set and let y be an element of S. Then $\{y\}$ is a trivial non empty subset of S.

We now state two propositions:

(6) For every non empty set S and for every element y of S such that $\{y\}$ is proper holds S is non trivial.

(7) For every non trivial non empty set S and for every element y of S holds $\{y\}$ is proper.

Let S be a trivial non empty set. Note that every non empty subset of S is non proper and every non empty subset of S which is non proper is also trivial.

Let S be a non trivial non empty set. Observe that every non empty subset of S which is trivial is also proper and every non empty subset of S which is non proper is also non trivial.

Let S be a non trivial non empty set. One can check that there exists a non empty subset of S which is trivial and proper and there exists a non empty subset of S which is non trivial and non proper.

One can prove the following propositions:

(8) Let Y be a 1-sorted structure and let y be an element of the carrier of Y. If $\{y\}$ is proper, then Y is non trivial.

(9) For every non trivial 1-sorted structure Y and for every element y of the carrier of Y holds $\{y\}$ is proper.

Let Y be a trivial 1-sorted structure. Note that every non empty subset of Y is non proper and every non empty subset of Y which is non proper is also trivial.

Let Y be a non trivial 1-sorted structure. One can verify that every non empty subset of Y which is trivial is also proper and every non empty subset of Y which is non proper is also non trivial.

Let Y be a non trivial 1-sorted structure. One can check that there exists a non empty subset of Y which is trivial and proper and there exists a non empty subset of Y which is non trivial and non proper.

2. Proper Subspaces of Topological Spaces

The following three propositions are true:

(10) Let X be a topological structure and let X_0 be a subspace of X. Then the topological structure of X_0 is a strict subspace of X.

(11) Let X be a topological structure and let X_1, X_2 be subspaces of X. Suppose the carrier of X_1 = the carrier of X_2. Then the topological structure of X_1 = the topological structure of X_2.

(12) Let Y_0, Y_1 be topological structures. Suppose the topological structure of Y_0 = the topological structure of Y_1. If Y_0 is topological space-like, then Y_1 is topological space-like.

Let Y be a topological structure. A subspace of Y is proper if:

(Def.3) For every subset A of Y such that A = the carrier of it holds A is proper.

We now state three propositions:

(13) Let Y_0 be a subspace of Y and let A be a subset of Y. If A = the carrier of Y_0, then A is proper iff Y_0 is proper.

(14) Let Y_0, Y_1 be subspaces of Y. Suppose the topological structure of Y_0 = the topological structure of Y_1. If Y_0 is proper, then Y_1 is proper.

(15) For every subspace Y_0 of Y such that the carrier of Y_0 = the carrier of Y holds Y_0 is non proper.

Let Y be a trivial topological structure. Observe that every subspace of Y is non proper and every subspace of Y which is non proper is also trivial.

Let Y be a non trivial topological structure. Observe that every subspace of Y which is trivial is also proper and every subspace of Y which is non proper is also non trivial.

Let Y be a topological structure. Observe that there exists a subspace of Y which is non proper and strict.

Next we state the proposition

(16) For every non proper subspace Y_0 of Y holds the topological structure of Y_0 = the topological structure of Y.

Let Y be a topological structure. One can check the following observations:

* every subspace of Y which is discrete is also topological space-like,
* every subspace of Y which is anti-discrete is also topological space-like,
* every subspace of Y which is non topological space-like is also non discrete, and
* every subspace of Y which is non topological space-like is also non anti-discrete.

One can prove the following propositions:

(17) Let Y_0, Y_1 be topological structures. Suppose the topological structure of Y_0 = the topological structure of Y_1. If Y_0 is discrete, then Y_1 is discrete.

(18) Let Y_0, Y_1 be topological structures. Suppose the topological structure of Y_0 = the topological structure of Y_1. If Y_0 is anti-discrete, then Y_1 is anti-discrete.

Let Y be a topological structure. One can verify the following observations:

* every subspace of Y which is discrete is also almost discrete,
* every subspace of Y which is non almost discrete is also non discrete,
* every subspace of Y which is anti-discrete is also almost discrete, and
* every subspace of Y which is non almost discrete is also non anti-discrete.

One can prove the following proposition

(19) Let Y_0, Y_1 be topological structures. Suppose the topological structure of Y_0 = the topological structure of Y_1. If Y_0 is almost discrete, then Y_1 is almost discrete.

Let Y be a topological structure. One can check the following observations:

* every subspace of Y which is discrete and anti-discrete is also trivial,
* every subspace of Y which is anti-discrete and non trivial is also non discrete, and
* every subspace of Y which is discrete and non trivial is also non anti-discrete.

Let Y be a topological structure and let y be a point of Y. The functor Sspace(y) yielding a strict subspace of Y is defined as follows:

(Def.4) The carrier of Sspace(y) = $\{y\}$.

Let Y be a topological structure. Observe that there exists a subspace of Y which is trivial and strict.

Let Y be a topological structure and let y be a point of Y. Then Sspace(y) is a trivial strict subspace of Y.

We now state three propositions:

(20) For every topological structure Y and for every point y of Y holds Sspace(y) is proper iff $\{y\}$ is proper.

(21) For every topological structure Y and for every point y of Y such that Sspace(y) is proper holds Y is non trivial.

(22) For every non trivial topological structure Y and for every point y of Y holds Sspace(y) is proper.

Let Y be a non trivial topological structure. One can verify that there exists a subspace of Y which is proper trivial and strict.

We now state two propositions:

(23) Let Y be a topological structure and let Y_0 be a trivial subspace of Y. Suppose Y_0 is topological space-like. Then there exists a point y of Y such that the topological structure of Y_0 = the topological structure of Sspace(y).

(24) Let Y be a topological structure and let y be a point of Y. If Sspace(y) is topological space-like, then Sspace(y) is discrete and anti-discrete.

Let Y be a topological structure. Note that every subspace of Y which is trivial and topological space-like is also discrete and anti-discrete.

Let X be a topological space. Note that there exists a subspace of X which is trivial strict and topological space-like.

Let X be a topological space and let x be a point of X. Then Sspace(x) is a trivial strict topological space-like subspace of X.

Let X be a topological space. Observe that there exists a subspace of X which is discrete anti-discrete and strict.

Let X be a topological space and let x be a point of X. Then Sspace(x) is a discrete anti-discrete strict subspace of X.

Let X be a topological space. One can check the following observations:

* every subspace of X which is non proper is also open and closed,
* every subspace of X which is non open is also proper, and
* every subspace of X which is non closed is also proper.

Let X be a topological space. Note that there exists a subspace of X which is open closed and strict.

Let X be a discrete topological space. Note that every subspace of X which is anti-discrete is also trivial and every subspace of X which is non trivial is also non anti-discrete.

Let X be a discrete non trivial topological space. Observe that there exists a subspace of X which is discrete open closed proper and strict.

Let X be an anti-discrete topological space. One can check that every subspace of X which is discrete is also trivial and every subspace of X which is non trivial is also non discrete.

Let X be an anti-discrete non trivial topological space. One can verify that every proper subspace of X is non open and non closed and every discrete subspace of X is trivial and proper.

Let X be an anti-discrete non trivial topological space. One can check that there exists a subspace of X which is anti-discrete non open non closed proper and strict.

Let X be an almost discrete non trivial topological space. Observe that there exists a subspace of X which is almost discrete proper and strict.

3. Maximal Discrete Subsets and Subspaces

Let Y be a topological structure. A subset of Y is discrete if:

(Def.5) For every subset D of Y such that $D \subseteq$ it there exists a subset G of Y such that G is open and it $\cap G = D$.

Let Y be a topological structure. Let us observe that a subset of Y is discrete if:

(Def.6) For every subset D of Y such that $D \subseteq$ it there exists a subset F of Y such that F is closed and it $\cap F = D$.

We now state three propositions:

(25) Let Y_0, Y_1 be topological structures, and let D_0 be a subset of Y_0, and let D_1 be a subset of Y_1. Suppose the topological structure of Y_0 = the topological structure of Y_1 and $D_0 = D_1$. If D_0 is discrete, then D_1 is discrete.

(26) Let Y be a topological structure, and let Y_0 be a subspace of Y, and let A be a subset of Y. Suppose A = the carrier of Y_0. Then A is discrete if and only if Y_0 is discrete.

(27) Let Y be a topological structure and let A be a subset of Y. Suppose A = the carrier of Y. Then A is discrete if and only if Y is discrete.

In the sequel Y will denote a topological structure.

We now state several propositions:

(28) For all subsets A, B of Y such that $B \subseteq A$ holds if A is discrete, then B is discrete.

(29) For all subsets A, B of Y such that A is discrete or B is discrete holds $A \cap B$ is discrete.

(30) Suppose that for all subsets P, Q of Y such that P is open and Q is open holds $P \cap Q$ is open and $P \cup Q$ is open. Let A, B be subsets of Y. Suppose A is open and B is open. If A is discrete and B is discrete, then $A \cup B$ is discrete.

(31) Suppose that for all subsets P, Q of Y such that P is closed and Q is closed holds $P \cap Q$ is closed and $P \cup Q$ is closed. Let A, B be subsets of Y. Suppose A is closed and B is closed. If A is discrete and B is discrete, then $A \cup B$ is discrete.

(32) Let A be a subset of Y. Suppose A is discrete. Let x be a point of Y. If $x \in A$, then there exists a subset G of Y such that G is open and $A \cap G = \{x\}$.

(33) Let A be a subset of Y. Suppose A is discrete. Let x be a point of Y. If $x \in A$, then there exists a subset F of Y such that F is closed and $A \cap F = \{x\}$.

In the sequel X denotes a topological space.

The following propositions are true:

(34) Let A_0 be a non empty subset of X. Suppose A_0 is discrete. Then there exists a discrete strict subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(35) Every empty subset of X is discrete.

(36) For every point x of X holds $\{x\}$ is discrete.

(37) Let A be a subset of X. Suppose that for every point x of X such that $x \in A$ there exists a subset G of X such that G is open and $A \cap G = \{x\}$. Then A is discrete.

(38) Let A, B be subsets of X. Suppose A is open and B is open. If A is discrete and B is discrete, then $A \cup B$ is discrete.

(39) Let A, B be subsets of X. Suppose A is closed and B is closed. If A is discrete and B is discrete, then $A \cup B$ is discrete.

(40) For every subset A of X such that A is everywhere dense holds if A is discrete, then A is open.

(41) For every subset A of X holds A is discrete iff for every subset D of X such that $D \subseteq A$ holds $A \cap \overline{D} = D$.

(42) For every subset A of X such that A is discrete and for every point x of X such that $x \in A$ holds $A \cap \overline{\{x\}} = \{x\}$.

(43) For every discrete topological space X holds every subset of X is discrete.

(44) Let X be an anti-discrete topological space and let A be a non empty subset of X. Then A is discrete if and only if A is trivial.

Let Y be a topological structure. A subset of Y is maximal discrete if:

(Def.7) It is discrete and for every subset D of Y such that D is discrete and it $\subseteq D$ holds it $= D$.

The following proposition is true

(45) Let Y_0, Y_1 be topological structures, and let D_0 be a subset of Y_0, and let D_1 be a subset of Y_1. Suppose the topological structure of $Y_0 =$ the topological structure of Y_1 and $D_0 = D_1$. If D_0 is maximal discrete, then D_1 is maximal discrete.

In the sequel X will denote a topological space.

Next we state several propositions:

(46) Every empty subset of X is not maximal discrete.

(47) For every subset A of X such that A is open holds if A is maximal discrete, then A is dense.

(48) For every subset A of X such that A is dense holds if A is discrete, then A is maximal discrete.

(49) Let X be a discrete topological space and let A be a subset of X. Then A is maximal discrete if and only if A is non proper.

(50) Let X be an anti-discrete topological space and let A be a non empty subset of X. Then A is maximal discrete if and only if A is trivial.

Let Y be a topological structure. A subspace of Y is maximal discrete if:

(Def.8) For every subset A of Y such that $A =$ the carrier of it holds A is maximal discrete.

One can prove the following proposition

(51) Let Y be a topological structure, and let Y_0 be a subspace of Y, and let A be a subset of Y. Suppose $A =$ the carrier of Y_0. Then A is maximal discrete if and only if Y_0 is maximal discrete.

Let Y be a topological structure. Note that every subspace of Y which is maximal discrete is also discrete and every subspace of Y which is non discrete is also non maximal discrete.

Next we state two propositions:

(52) Let X_0 be a subspace of X. Then X_0 is maximal discrete if and only if the following conditions are satisfied:

(i) X_0 is discrete, and

(ii) for every discrete subspace Y_0 of X such that X_0 is a subspace of Y_0 holds the topological structure of $X_0 =$ the topological structure of Y_0.

(53) Let A_0 be a non empty subset of X. Suppose A_0 is maximal discrete. Then there exists a strict subspace X_0 of X such that X_0 is maximal discrete and $A_0 =$ the carrier of X_0.

Let X be a discrete topological space. One can verify the following observations:

* every subspace of X which is maximal discrete is also non proper,
* every subspace of X which is proper is also non maximal discrete,
* every subspace of X which is non proper is also maximal discrete, and
* every subspace of X which is non maximal discrete is also proper.

Let X be an anti-discrete topological space. One can check the following observations:

* every subspace of X which is maximal discrete is also trivial,
* every subspace of X which is non trivial is also non maximal discrete,
* every subspace of X which is trivial is also maximal discrete, and

* every subspace of X which is non maximal discrete is also non trivial.

4. Maximal Discrete Subspaces of Almost Discrete Spaces

The scheme *ExChoiceFCol* deals with a topological structure $\mathcal{A}$, a family $\mathcal{B}$ of subsets of $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a function f from $\mathcal{B}$ into the carrier of $\mathcal{A}$ such that for every subset S of $\mathcal{A}$ such that $S \in \mathcal{B}$ holds $\mathcal{P}[S, f(S)]$

provided the following condition is met:

- For every subset S of $\mathcal{A}$ such that $S \in \mathcal{B}$ there exists a point x of $\mathcal{A}$ such that $\mathcal{P}[S, x]$.

In the sequel X will denote an almost discrete topological space.

We now state a number of propositions:

(54) For every subset A of X holds $\overline{A} = \bigcup\{\overline{\{a\}} : a$ ranges over points of X, $a \in A\}$.

(55) For all points a, b of X such that $a \in \overline{\{b\}}$ holds $\overline{\{a\}} = \overline{\{b\}}$.

(56) For all points a, b of X holds $\overline{\{a\}} \cap \overline{\{b\}} = \emptyset$ or $\overline{\{a\}} = \overline{\{b\}}$.

(57) Let A be a subset of X. Suppose that for every point x of X such that $x \in A$ there exists a subset F of X such that F is closed and $A \cap F = \{x\}$. Then A is discrete.

(58) For every subset A of X such that for every point x of X such that $x \in A$ holds $A \cap \overline{\{x\}} = \{x\}$ holds A is discrete.

(59) Let A be a subset of X. Then A is discrete if and only if for all points a, b of X such that $a \in A$ and $b \in A$ holds if $a \neq b$, then $\overline{\{a\}} \cap \overline{\{b\}} = \emptyset$.

(60) Let A be a subset of X. Then A is discrete if and only if for every point x of X such that $x \in \overline{A}$ there exists a point a of X such that $a \in A$ and $A \cap \overline{\{x\}} = \{a\}$.

(61) For every subset A of X such that A is open or closed holds if A is maximal discrete, then A is not proper.

(62) For every subset A of X such that A is maximal discrete holds A is dense.

(63) For every subset A of X such that A is maximal discrete holds $\bigcup\{\overline{\{a\}} : a$ ranges over points of X, $a \in A\} =$ the carrier of X.

(64) Let A be a subset of X. Then A is maximal discrete if and only if for every point x of X there exists a point a of X such that $a \in A$ and $A \cap \overline{\{x\}} = \{a\}$.

(65) For every subset A of X such that A is discrete there exists a subset M of X such that $A \subseteq M$ and M is maximal discrete.

(66) There exists subset of X which is maximal discrete.

(67) Let Y_0 be a discrete subspace of X. Then there exists a strict subspace X_0 of X such that Y_0 is a subspace of X_0 and X_0 is maximal discrete.

Let X be an almost discrete non discrete topological space. One can verify that every subspace of X which is maximal discrete is also proper and every subspace of X which is non proper is also non maximal discrete.

Let X be an almost discrete non anti-discrete topological space. Observe that every subspace of X which is maximal discrete is also non trivial and every subspace of X which is trivial is also non maximal discrete.

Let X be an almost discrete topological space. Note that there exists a subspace of X which is maximal discrete and strict.

5. Continuous Mappings and Almost Discrete Spaces

The scheme *MapExChoiceF* concerns a topological structure $\mathcal{A}$, a topological structure $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a map f from $\mathcal{A}$ into $\mathcal{B}$ such that for every point x of $\mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the parameters have the following property:

- For every point x of $\mathcal{A}$ there exists a point y of $\mathcal{B}$ such that $\mathcal{P}[x, y]$.

In the sequel X, Y are topological spaces.

Next we state four propositions:

(68) For every discrete topological space X holds every mapping from X into Y is continuous.

(69) If for every topological space Y holds every mapping from X into Y is continuous, then X is discrete.

(70) For every anti-discrete topological space Y holds every mapping from X into Y is continuous.

(71) If for every topological space X holds every mapping from X into Y is continuous, then Y is anti-discrete.

In the sequel X will be a discrete topological space and X_0 will be a subspace of X.

One can prove the following two propositions:

(72) There exists continuous mapping from X into X_0 which is a retraction.

(73) X_0 is a retract of X.

In the sequel X will be an almost discrete topological space and X_0 will be a maximal discrete subspace of X.

Next we state four propositions:

(74) There exists continuous mapping from X into X_0 which is a retraction.

(75) X_0 is a retract of X.

(76) Let r be a continuous mapping from X into X_0. Suppose r is a retraction. Let F be a subset of X_0 and let E be a subset of X. If $F = E$, then $r^{-1} F = \overline{E}$.

(77) Let r be a continuous mapping from X into X_0. Suppose r is a retraction. Let a be a point of X_0 and let b be a point of X. If $a = b$, then $r^{-1}\{a\} = \overline{\{b\}}$.

In the sequel X_0 is a discrete subspace of X.

The following two propositions are true:

(78) There exists continuous mapping from X into X_0 which is a retraction.

(79) X_0 is a retract of X.

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On Nowhere and Everywhere Dense Subspaces of Topological Spaces

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Summary. Let X be a topological space and let X_0 be a subspace of X with the carrier A. X_0 is called *boundary* (*dense*) in X if A is boundary (dense), i.e., $\operatorname{Int} A = \emptyset$ ($\overline{A} =$ the carrier of X); X_0 is called *nowhere dense* (*everywhere dense*) in X if A is nowhere dense (everywhere dense), i.e., $\operatorname{Int} \overline{A} = \emptyset$ ($\overline{\operatorname{Int} A} =$ the carrier of X) (see [5] and comp. [8]).

Our purpose is to list, using Mizar formalism, a number of properties of such subspaces, mostly in non-discrete (non-almost-discrete) spaces (comp. [5]). Recall that X is called *discrete* if every subset of X is open (closed); X is called *almost discrete* if every open subset of X is closed; equivalently, if every closed subset of X is open (see [1], [4] and comp. [8],[7]). We have the following characterization of non-discrete spaces: *X is non-discrete iff there exists a boundary subspace in X*. Hence, *X is non-discrete iff there exists a dense proper subspace in X*. We have the following analogous characterization of non-almost-discrete spaces: *X is non-almost-discrete iff there exists a nowhere dense subspace in X*. Hence, *X is non-almost-discrete iff there exists an everywhere dense proper subspace in X*.

Note that some interdependencies between boundary, dense, nowhere and everywhere dense subspaces are also indicated. These have the form of observations in the text and they correspond to the existential and to the conditional clusters in the Mizar System. These clusters guarantee the existence and ensure the extension of types supported automatically by the Mizar System.

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The terminology and notation used in this paper have been introduced in the following articles: [11], [9], [12], [10], [6], [3], [1], [5], and [2].

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1. Some Properties of Subsets of a Topological Space

In the sequel X denotes a topological space and A, B denote subsets of X. The following propositions are true:

(1) If A and B constitute a decomposition, then A is non empty iff B is proper.

(2) If A and B constitute a decomposition, then A is dense iff B is boundary.

(3) If A and B constitute a decomposition, then A is boundary iff B is dense.

(4) If A and B constitute a decomposition, then A is everywhere dense iff B is nowhere dense.

(5) If A and B constitute a decomposition, then A is nowhere dense iff B is everywhere dense.

In the sequel Y_1, Y_2 will be subspaces of X.

Next we state three propositions:

(6) If Y_1 and Y_2 constitute a decomposition, then Y_1 is proper and Y_2 is proper.

(7) Let X be a non trivial topological space and let D be a non empty proper subset of X. Then there exists a proper strict subspace Y_0 of X such that $D =$ the carrier of Y_0.

(8) Let X be a non trivial topological space and let Y_1 be a proper subspace of X. Then there exists a proper strict subspace Y_2 of X such that Y_1 and Y_2 constitute a decomposition.

2. Dense and Everywhere Dense Subspaces

Let X be a topological space. A subspace of X is dense if:

(Def.1) For every subset A of X such that $A =$ the carrier of it holds A is dense.

The following proposition is true

(9) Let X_0 be a subspace of X and let A be a subset of X. If $A =$ the carrier of X_0, then X_0 is dense iff A is dense.

Let X be a topological space. One can check the following observations:

* every subspace of X which is dense and closed is also non proper,
* every subspace of X which is dense and proper is also non closed, and
* every subspace of X which is proper and closed is also non dense.

Let X be a topological space. Note that there exists a subspace of X which is dense and strict.

We now state several propositions:

(10) Let A_0 be a non empty subset of X. Suppose A_0 is dense. Then there exists a dense strict subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(11) Let X_0 be a dense subspace of X, and let A be a subset of X, and let B be a subset of X_0. If $A = B$, then B is dense iff A is dense.

(12) For every dense subspace X_1 of X and for every subspace X_2 of X such that X_1 is a subspace of X_2 holds X_2 is dense.

(13) Let X_1 be a dense subspace of X and let X_2 be a subspace of X. If X_1 is a subspace of X_2, then X_1 is a dense subspace of X_2.

(14) For every dense subspace X_1 of X holds every dense subspace of X_1 is a dense subspace of X.

(15) Let Y_1, Y_2 be topological spaces. Suppose $Y_2 =$ the topological structure of Y_1. Then Y_1 is a dense subspace of X if and only if Y_2 is a dense subspace of X.

Let X be a topological space. A subspace of X is everywhere dense if:

(Def.2) For every subset A of X such that $A =$ the carrier of it holds A is everywhere dense.

Next we state the proposition

(16) Let X_0 be a subspace of X and let A be a subset of X. Suppose $A =$ the carrier of X_0. Then X_0 is everywhere dense if and only if A is everywhere dense.

Let X be a topological space. One can check the following observations:

* every subspace of X which is everywhere dense is also dense,
* every subspace of X which is non dense is also non everywhere dense,
* every subspace of X which is non proper is also everywhere dense, and
* every subspace of X which is non everywhere dense is also proper.

Let X be a topological space. Observe that there exists a subspace of X which is everywhere dense and strict.

We now state several propositions:

(17) Let A_0 be a non empty subset of X. Suppose A_0 is everywhere dense. Then there exists an everywhere dense strict subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(18) Let X_0 be an everywhere dense subspace of X, and let A be a subset of X, and let B be a subset of X_0. Suppose $A = B$. Then B is everywhere dense if and only if A is everywhere dense.

(19) Let X_1 be an everywhere dense subspace of X and let X_2 be a subspace of X. If X_1 is a subspace of X_2, then X_2 is everywhere dense.

(20) Let X_1 be an everywhere dense subspace of X and let X_2 be a subspace of X. Suppose X_1 is a subspace of X_2. Then X_1 is an everywhere dense subspace of X_2.

(21) For every everywhere dense subspace X_1 of X holds every everywhere dense subspace of X_1 is an everywhere dense subspace of X.

(22) Let Y_1, Y_2 be topological spaces. Suppose $Y_2 =$ the topological structure of Y_1. Then Y_1 is an everywhere dense subspace of X if and only if Y_2 is an everywhere dense subspace of X.

Let X be a topological space. One can check the following observations:

* every subspace of X which is dense and open is also everywhere dense,
* every subspace of X which is dense and non everywhere dense is also non open, and
* every subspace of X which is open and non everywhere dense is also non dense.

Let X be a topological space. Note that there exists a subspace of X which is dense open and strict.

We now state two propositions:

(23) Let A_0 be a non empty subset of X. Suppose A_0 is dense and open. Then there exists a dense open strict subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(24) For every subspace X_0 of X holds X_0 is everywhere dense iff there exists dense open strict subspace of X which is a subspace of X_0.

In the sequel X_1, X_2 denote subspaces of X.

One can prove the following four propositions:

(25) If X_1 is dense or X_2 is dense, then $X_1 \cup X_2$ is a dense subspace of X.

(26) If X_1 is everywhere dense or X_2 is everywhere dense, then $X_1 \cup X_2$ is an everywhere dense subspace of X.

(27) If X_1 is everywhere dense and X_2 is everywhere dense, then $X_1 \cap X_2$ is an everywhere dense subspace of X.

(28) Suppose X_1 is everywhere dense and X_2 is dense or X_1 is dense and X_2 is everywhere dense. Then $X_1 \cap X_2$ is a dense subspace of X.

3. Boundary and Nowhere Dense Subspaces

Let X be a topological space. A subspace of X is boundary if:

(Def.3) For every subset A of X such that $A =$ the carrier of it holds A is boundary.

We now state the proposition

(29) Let X_0 be a subspace of X and let A be a subset of X. Suppose $A =$ the carrier of X_0. Then X_0 is boundary if and only if A is boundary.

Let X be a topological space. One can verify the following observations:

* every subspace of X which is open is also non boundary,
* every subspace of X which is boundary is also non open,
* every subspace of X which is everywhere dense is also non boundary, and
* every subspace of X which is boundary is also non everywhere dense.

Next we state several propositions:

(30) Let A_0 be a non empty subset of X. Suppose A_0 is boundary. Then there exists a strict subspace X_0 of X such that X_0 is boundary and $A_0 =$ the carrier of X_0.

(31) Let X_1, X_2 be subspaces of X. Suppose X_1 and X_2 constitute a decomposition. Then X_1 is dense if and only if X_2 is boundary.

(32) Let X_1, X_2 be subspaces of X. Suppose X_1 and X_2 constitute a decomposition. Then X_1 is boundary if and only if X_2 is dense.

(33) Let X_0 be a subspace of X. Suppose X_0 is boundary. Let A be a subset of X. If $A \subseteq$ the carrier of X_0, then A is boundary.

(34) For all subspaces X_1, X_2 of X such that X_1 is boundary holds if X_2 is a subspace of X_1, then X_2 is boundary.

Let X be a topological space. A subspace of X is nowhere dense if:

(Def.4) For every subset A of X such that $A =$ the carrier of it holds A is nowhere dense.

We now state the proposition

(35) Let X_0 be a subspace of X and let A be a subset of X. Suppose $A =$ the carrier of X_0. Then X_0 is nowhere dense if and only if A is nowhere dense.

Let X be a topological space. One can verify the following observations:

* every subspace of X which is nowhere dense is also boundary,
* every subspace of X which is non boundary is also non nowhere dense,
* every subspace of X which is nowhere dense is also non dense, and
* every subspace of X which is dense is also non nowhere dense.

In the sequel X will denote a topological space.

One can prove the following propositions:

(36) Let A_0 be a non empty subset of X. Suppose A_0 is nowhere dense. Then there exists a strict subspace X_0 of X such that X_0 is nowhere dense and $A_0 =$ the carrier of X_0.

(37) Let X_1, X_2 be subspaces of X. Suppose X_1 and X_2 constitute a decomposition. Then X_1 is everywhere dense if and only if X_2 is nowhere dense.

(38) Let X_1, X_2 be subspaces of X. Suppose X_1 and X_2 constitute a decomposition. Then X_1 is nowhere dense if and only if X_2 is everywhere dense.

(39) Let X_0 be a subspace of X. Suppose X_0 is nowhere dense. Let A be a subset of X. If $A \subseteq$ the carrier of X_0, then A is nowhere dense.

(40) Let X_1, X_2 be subspaces of X. Suppose X_1 is nowhere dense. If X_2 is a subspace of X_1, then X_2 is nowhere dense.

Let X be a topological space. One can verify the following observations:

* every subspace of X which is boundary and closed is also nowhere dense,
* every subspace of X which is boundary and non nowhere dense is also non closed, and

* every subspace of X which is closed and non nowhere dense is also non boundary.

The following propositions are true:

(41) Let A_0 be a non empty subset of X. Suppose A_0 is boundary and closed. Then there exists a closed strict subspace X_0 of X such that X_0 is boundary and $A_0 =$ the carrier of X_0.

(42) Let X_0 be a subspace of X. Then X_0 is nowhere dense if and only if there exists a closed strict subspace X_1 of X such that X_1 is boundary and X_0 is a subspace of X_1.

In the sequel X_1, X_2 will be subspaces of X.

One can prove the following propositions:

(43) If X_1 is boundary or X_2 is boundary and if X_1 meets X_2, then $X_1 \cap X_2$ is boundary.

(44) If X_1 is nowhere dense and X_2 is nowhere dense, then $X_1 \cup X_2$ is nowhere dense.

(45) If X_1 is nowhere dense and X_2 is boundary or X_1 is boundary and X_2 is nowhere dense, then $X_1 \cup X_2$ is boundary.

(46) If X_1 is nowhere dense or X_2 is nowhere dense and if X_1 meets X_2, then $X_1 \cap X_2$ is nowhere dense.

4. Dense and Boundary Subspaces of Non-discrete Spaces

Next we state two propositions:

(47) For every topological space X such that every subspace of X is non boundary holds X is discrete.

(48) For every non trivial topological space X such that every proper subspace of X is non dense holds X is discrete.

Let X be a discrete topological space. One can check the following observations:

* every subspace of X is non boundary,
* every subspace of X which is proper is also non dense, and
* every subspace of X which is dense is also non proper.

Let X be a discrete topological space. Observe that there exists a subspace of X which is non boundary and strict.

Let X be a discrete non trivial topological space. Note that there exists a subspace of X which is non dense and strict.

One can prove the following two propositions:

(49) For every topological space X such that there exists subspace of X which is boundary holds X is non discrete.

(50) For every topological space X such that there exists subspace of X which is dense and proper holds X is non discrete.

Let X be a non discrete topological space. One can check that there exists a subspace of X which is boundary and strict and there exists a subspace of X which is dense proper and strict.

In the sequel X will be a non discrete topological space.

We now state several propositions:

(51) Let A_0 be a non empty subset of X. Suppose A_0 is boundary. Then there exists a boundary strict subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(52) Let A_0 be a non empty proper subset of X. Suppose A_0 is dense. Then there exists a dense proper strict subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(53) Let X_1 be a boundary subspace of X. Then there exists a dense proper strict subspace X_2 of X such that X_1 and X_2 constitute a decomposition.

(54) Let X_1 be a dense proper subspace of X. Then there exists a boundary strict subspace X_2 of X such that X_1 and X_2 constitute a decomposition.

(55) Let Y_1, Y_2 be topological spaces. Suppose $Y_2 =$ the topological structure of Y_1. Then Y_1 is a boundary subspace of X if and only if Y_2 is a boundary subspace of X.

5. Everywhere and Nowhere Dense Subspaces of Non-almost-discrete Spaces

Next we state two propositions:

(56) For every topological space X such that every subspace of X is non nowhere dense holds X is almost discrete.

(57) For every non trivial topological space X such that every proper subspace of X is non everywhere dense holds X is almost discrete.

Let X be an almost discrete topological space. One can verify the following observations:

* every subspace of X is non nowhere dense,
* every subspace of X which is proper is also non everywhere dense,
* every subspace of X which is everywhere dense is also non proper,
* every subspace of X which is boundary is also non closed,
* every subspace of X which is closed is also non boundary,
* every subspace of X which is dense and proper is also non open,
* every subspace of X which is dense and open is also non proper, and
* every subspace of X which is open and proper is also non dense.

Let X be an almost discrete topological space. One can verify that there exists a subspace of X which is non nowhere dense and strict.

Let X be an almost discrete non trivial topological space. Note that there exists a subspace of X which is non everywhere dense and strict.

The following four propositions are true:

(58) For every topological space X such that there exists subspace of X which is nowhere dense holds X is non almost discrete.

(59) For every topological space X such that there exists subspace of X which is boundary and closed holds X is non almost discrete.

(60) For every topological space X such that there exists subspace of X which is everywhere dense and proper holds X is non almost discrete.

(61) For every topological space X such that there exists subspace of X which is dense and open and proper holds X is non almost discrete.

Let X be a non almost discrete topological space. One can check that there exists a subspace of X which is nowhere dense and strict and there exists a subspace of X which is everywhere dense proper and strict.

In the sequel X denotes a non almost discrete topological space.

The following propositions are true:

(62) Let A_0 be a non empty subset of X. Suppose A_0 is nowhere dense. Then there exists a nowhere dense strict subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(63) Let A_0 be a non empty proper subset of X. Suppose A_0 is everywhere dense. Then there exists an everywhere dense proper strict subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(64) Let X_1 be a nowhere dense subspace of X. Then there exists an everywhere dense proper strict subspace X_2 of X such that X_1 and X_2 constitute a decomposition.

(65) Let X_1 be an everywhere dense proper subspace of X. Then there exists a nowhere dense strict subspace X_2 of X such that X_1 and X_2 constitute a decomposition.

(66) Let Y_1, Y_2 be topological spaces. Suppose $Y_2 =$ the topological structure of Y_1. Then Y_1 is a nowhere dense subspace of X if and only if Y_2 is a nowhere dense subspace of X.

Let X be a non almost discrete topological space. One can verify that there exists a subspace of X which is boundary closed and strict and there exists a subspace of X which is dense open proper and strict.

Next we state several propositions:

(67) Let A_0 be a non empty subset of X. Suppose A_0 is boundary and closed. Then there exists a boundary closed strict subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(68) Let A_0 be a non empty proper subset of X. Suppose A_0 is dense and open. Then there exists a dense open proper strict subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(69) Let X_1 be a boundary closed subspace of X. Then there exists a dense open proper strict subspace X_2 of X such that X_1 and X_2 constitute a decomposition.

(70) Let X_1 be a dense open proper subspace of X. Then there exists a boundary closed strict subspace X_2 of X such that X_1 and X_2 constitute a decomposition.

(71) Let X_0 be a subspace of X. Then X_0 is nowhere dense if and only if there exists a boundary closed strict subspace X_1 of X such that X_0 is a subspace of X_1.

(72) Let X_0 be a nowhere dense subspace of X. Then

(i) X_0 is boundary or closed, or

(ii) there exists an everywhere dense proper strict subspace X_1 of X and there exists a boundary closed strict subspace X_2 of X such that $X_1 \cap X_2 =$ the topological structure of X_0 and $X_1 \cup X_2 =$ the topological structure of X.

(73) Let X_0 be an everywhere dense subspace of X. Then

(i) X_0 is dense or open, or

(ii) there exists a dense open proper strict subspace X_1 of X and there exists a nowhere dense strict subspace X_2 of X such that X_1 misses X_2 and $X_1 \cup X_2 =$ the topological structure of X_0.

(74) Let X_0 be a nowhere dense subspace of X. Then there exists a dense open proper strict subspace X_1 of X and there exists a boundary closed strict subspace X_2 of X such that X_1 and X_2 constitute a decomposition and X_0 is a subspace of X_2.

(75) Let X_0 be an everywhere dense proper subspace of X. Then there exists a dense open proper strict subspace X_1 of X and there exists a boundary closed strict subspace X_2 of X such that X_1 and X_2 constitute a decomposition and X_1 is a subspace of X_0.

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