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The Product and the Determinant of Matrices with Entries in a Field

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Summary. Concerned with a generalization of concepts introduced in [17], i.e. there are introduced the sum and the product of matrices of any dimension of elements of any field.

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The articles [15], [28], [10], [11], [5], [7], [6], [12], [16], [20], [27], [19], [23], [13], [9], [8], [21], [26], [1], [17], [25], [18], [4], [3], [24], [29], [2], [22], and [14] provide the notation and terminology for this paper.

For simplicity we follow a convention: i, j, k, l, n, m denote natural numbers, I, J, D denote non empty sets, K denotes a field, a denotes an element of D, and p, q denote finite sequences of elements of D.

We now state two propositions:

(1) If $n = n + k$, then $k = 0$.

(2) For every natural number n holds $n = 0$ or $n = 1$ or $n = 2$ or $n > 2$.

In the sequel A, B will denote matrices over K of dimension $n \times m$.

Let us consider K, n, m. The functor $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n\times m}$ yields a matrix over K of dimension $n \times m$ and is defined as follows:

(Def.1) $$\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n\times m} = n \longmapsto (m \longmapsto 0_K).$$

Let us consider K and let A be a matrix over K. The functor $-A$ yields a matrix over K and is defined by:

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(Def.2) $\operatorname{len}(-A) = \operatorname{len} A$ and $\operatorname{width}(-A) = \operatorname{width} A$ and for all i, j such that $\langle i, j \rangle \in$ the indices of A holds $(-A)_{i,j} = -A_{i,j}$.

Let us consider K and let A, B be matrices over K. Let us assume that $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$. The functor $A + B$ yielding a matrix over K is defined as follows:

(Def.3) $\operatorname{len}(A + B) = \operatorname{len} A$ and $\operatorname{width}(A + B) = \operatorname{width} A$ and for all i, j such that $\langle i, j \rangle \in$ the indices of A holds $(A + B)_{i,j} = A_{i,j} + B_{i,j}$.

The following proposition is true

(3) For all i, j such that $\langle i, j \rangle \in$ the indices of $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n \times m}$ holds $(\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n \times m})_{i,j} = 0_K$.

In the sequel A, B denote matrices over K.

The following propositions are true:

(4) For all matrices A, B over K such that $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ holds $A + B = B + A$.

(5) For all matrices A, B, C over K such that $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{len} A = \operatorname{len} C$ and $\operatorname{width} A = \operatorname{width} B$ and $\operatorname{width} A = \operatorname{width} C$ holds $(A+B)+C = A + (B + C)$.

(6) For every matrix A over K of dimension $n \times m$ holds $A + \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n \times m} = A$.

(7) For every matrix A over K of dimension $n \times m$ holds $A + -A = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n \times m}$

Let us consider K and let A, B be matrices over K. Let us assume that $\operatorname{width} A = \operatorname{len} B$. The functor $A \cdot B$ yields a matrix over K and is defined as follows:

(Def.4) $\operatorname{len}(A \cdot B) = \operatorname{len} A$ and $\operatorname{width}(A \cdot B) = \operatorname{width} B$ and for all i, j such that $\langle i, j \rangle \in$ the indices of $A \cdot B$ holds $(A \cdot B)_{i,j} = \operatorname{Line}(A, i) \cdot B_{\square, j}$.

Let us consider n, k, m, let us consider K, let A be a matrix over K of dimension $n \times k$, and let B be a matrix over K of dimension $\operatorname{width} A \times m$. Then $A \cdot B$ is a matrix over K of dimension $\operatorname{len} A \times \operatorname{width} B$.

Let us consider K, let M be a matrix over K, and let a be an element of the carrier of K. The functor $a \cdot M$ yields a matrix over K and is defined by:

(Def.5) $\mathrm{len}(a \cdot M) = \mathrm{len}\, M$ and $\mathrm{width}(a \cdot M) = \mathrm{width}\, M$ and for all i, j such that $\langle i, j \rangle \in$ the indices of M holds $(a \cdot M)_{i,j} = a \cdot M_{i,j}$.

Let us consider K, let M be a matrix over K, and let a be an element of the carrier of K. The functor $M \cdot a$ yields a matrix over K and is defined by:

(Def.6) $M \cdot a = a \cdot M$.

One can prove the following propositions:

(8) For all finite sequences p, q of elements of the carrier of K such that $\mathrm{len}\, p = \mathrm{len}\, q$ holds $\mathrm{len}(p \bullet q) = \mathrm{len}\, p$ and $\mathrm{len}(p \bullet q) = \mathrm{len}\, q$.

(9) For all i, l such that $\langle i, l \rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ and $l = i$ holds $\mathrm{Line}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, i)(l) = 1_K$.

(10) For all i, l such that $\langle i, l \rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ and $l \neq i$ holds $\mathrm{Line}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, i)(l) = 0_K$.

(11) For all l, j such that $\langle l, j \rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ and $l = j$ holds $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})_{\square, j}(l) = 1_K$.

(12) For all l, j such that $\langle l, j \rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ and $l \neq j$ holds $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})_{\square, j}(l) = 0_K$.

(13) $\sum(n \longmapsto 0_K) = 0_K$.

(14) Let p be a finite sequence of elements of the carrier of K and given i. Suppose $i \in \mathrm{Seg}\,\mathrm{len}\, p$ and for every k such that $k \in \mathrm{Seg}\,\mathrm{len}\, p$ and $k \neq i$ holds $p(k) = 0_K$. Then $\sum p = p(i)$.

(15) For all finite sequences p, q of elements of the carrier of K holds $\mathrm{len}(p \bullet$

$q) = \min(\operatorname{len} p, \operatorname{len} q)$.

(16) Let p, q be finite sequences of elements of the carrier of K and given i. Suppose $i \in \operatorname{Seg} \operatorname{len} p$ and $p(i) = 1_K$ and for every k such that $k \in \operatorname{Seg} \operatorname{len} p$ and $k \neq i$ holds $p(k) = 0_K$. Given j. Suppose $j \in \operatorname{Seg} \operatorname{len}(p \bullet q)$. Then if $i = j$, then $(p \bullet q)(j) = q(i)$ and if $i \neq j$, then $(p \bullet q)(j) = 0_K$.

(17) For all i, j such that $\langle i, j \rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ holds if $i = j$, then $\operatorname{Line}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, i)(j) = 1_K$ and if $i \neq j$, then $\operatorname{Line}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, i)(j) = 0_K$.

(18) For all i, j such that $\langle i, j \rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ holds if $i = j$, then $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})_{\square, j}(i) = 1_K$ and if $i \neq j$, then $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})_{\square, j}(i) = 0_K$.

(19) Let p, q be finite sequences of elements of the carrier of K and given i. Suppose $i \in \operatorname{Seg} \operatorname{len} p$ and $i \in \operatorname{Seg} \operatorname{len} q$ and $p(i) = 1_K$ and for every k such that $k \in \operatorname{Seg} \operatorname{len} p$ and $k \neq i$ holds $p(k) = 0_K$. Then $\sum(p \bullet q) = q(i)$.

(20) For every matrix A over K of dimension n holds $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n} \cdot A = A$.

(21) For every matrix A over K of dimension n holds $A \cdot \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n} = A$.

(22) For all elements a, b of the carrier of K holds $\langle\langle a \rangle\rangle \cdot \langle\langle b \rangle\rangle = \langle\langle a \cdot b \rangle\rangle$.

(23) For all elements a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 of the carrier of K holds $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 \cdot a_2 + b_1 \cdot c_2 & a_1 \cdot b_2 + b_1 \cdot d_2 \\ c_1 \cdot a_2 + d_1 \cdot c_2 & c_1 \cdot b_2 + d_1 \cdot d_2 \end{pmatrix}$.

(24) For all matrices A, B over K such that width $A = \operatorname{len} B$ and width $B \neq 0$ holds $(A \cdot B)^{\mathrm{T}} = B^{\mathrm{T}} \cdot A^{\mathrm{T}}$.

Let I, J be non empty sets, let X be an element of Fin I, and let Y be an element of Fin J. Then $[: X, Y :]$ is an element of Fin$[: I, J :]$.

Let I, J, D be non empty sets, let G be a binary operation on D, let f be a function from I into D, and let g be a function from J into D. Then $G \circ (f, g)$ is a function from $[: I, J :]$ into D.

The following propositions are true:

(25) Let I, J, D be non empty sets, and let F, G be binary operations on D, and let f be a function from I into D, and let g be a function from J into D, and let X be an element of Fin I, and let Y be an element of Fin J. Suppose F is commutative and associative but $[: Y, X :] \neq \emptyset$ or F has a unity but G is commutative. Then $F\text{-}\sum_{[: X, Y :]}(G \circ (f, g)) = F\text{-}\sum_{[: Y, X :]}(G \circ (g, f))$.

(26) Let I, J be non empty sets, and let F, G be binary operations on D, and let f be a function from I into D, and let g be a function from J into D. Suppose F is commutative and associative and has a unity. Let x be an element of I and let y be an element of J. Then $F\text{-}\sum_{[: \{x\}, \{y\} :]}(G \circ (f, g)) = F\text{-}\sum_{\{x\}} G^{\circ}(f, F\text{-}\sum_{\{y\}} g)$.

(27) Let I, J be non empty sets, and let F, G be binary operations on D, and let f be a function from I into D, and let g be a function from J into D, and let X be an element of Fin I, and let Y be an element of Fin J. Suppose F is commutative and associative and has a unity and an inverse operation and G is distributive w.r.t. F. Let x be an element of I. Then $F\text{-}\sum_{[: \{x\}, Y :]}(G \circ (f, g)) = F\text{-}\sum_{\{x\}} G^{\circ}(f, F\text{-}\sum_{Y} g)$.

(28) Let I, J be non empty sets, and let F, G be binary operations on D, and let f be a function from I into D, and let g be a function from J into D, and let X be an element of Fin I, and let Y be an element of Fin J. Suppose F is commutative and associative and has a unity and an inverse operation and G is distributive w.r.t. F. Then $F\text{-}\sum_{[: X, Y :]}(G \circ (f, g)) = F\text{-}\sum_{X} G^{\circ}(f, F\text{-}\sum_{Y} g)$.

(29) Let I, J be non empty sets, and let F, G be binary operations on D, and let f be a function from I into D, and let g be a function from J into D. Suppose F is commutative and associative and has a unity and G is commutative. Let x be an element of I and let y be an element of J. Then $F\text{-}\sum_{[: \{x\}, \{y\} :]}(G \circ (f, g)) = F\text{-}\sum_{\{y\}} G^{\circ}(F\text{-}\sum_{\{x\}} f, g)$.

(30) Let I, J be non empty sets, and let F, G be binary operations on D, and let f be a function from I into D, and let g be a function from J into D, and let X be an element of Fin I, and let Y be an element of Fin J. Suppose that

(i) F is commutative and associative and has a unity and an inverse operation, and

(ii) G is distributive w.r.t. F and commutative.

Then $F\text{-}\sum_{[: X, Y :]}(G \circ (f, g)) = F\text{-}\sum_{Y} G^{\circ}(F\text{-}\sum_{X} f, g)$.

(31) Let I, J be non empty sets, and let F be a binary operation on D, and let f be a function from $[:I, J:]$ into D, and let g be a function from I into D, and let Y be an element of $\mathrm{Fin}\, J$. Suppose F is commutative and associative and has a unity and an inverse operation. Let x be an element of I. If for every element i of I holds $g(i) = F\text{-}\sum_{Y}(\mathrm{curry}\, f)(i)$, then $F\text{-}\sum_{[:\{x\}, Y:]} f = F\text{-}\sum_{\{x\}} g$.

(32) Let I, J be non empty sets, and let F be a binary operation on D, and let f be a function from $[:I, J:]$ into D, and let g be a function from I into D, and let X be an element of $\mathrm{Fin}\, I$, and let Y be an element of $\mathrm{Fin}\, J$. Suppose for every element i of I holds $g(i) = F\text{-}\sum_{Y}(\mathrm{curry}\, f)(i)$ and F is commutative and associative and has a unity and an inverse operation. Then $F\text{-}\sum_{[:X, Y:]} f = F\text{-}\sum_{X} g$.

(33) Let I, J be non empty sets, and let F be a binary operation on D, and let f be a function from $[:I, J:]$ into D, and let g be a function from J into D, and let X be an element of $\mathrm{Fin}\, I$. Suppose F is commutative and associative and has a unity and an inverse operation. Let y be an element of J. If for every element j of J holds $g(j) = F\text{-}\sum_{X}(\mathrm{curry}'\, f)(j)$, then $F\text{-}\sum_{[:X, \{y\}:]} f = F\text{-}\sum_{\{y\}} g$.

(34) Let I, J be non empty sets, and let F be a binary operation on D, and let f be a function from $[:I, J:]$ into D, and let g be a function from J into D, and let X be an element of $\mathrm{Fin}\, I$, and let Y be an element of $\mathrm{Fin}\, J$. Suppose for every element j of J holds $g(j) = F\text{-}\sum_{X}(\mathrm{curry}'\, f)(j)$ and F is commutative and associative and has a unity and an inverse operation. Then $F\text{-}\sum_{[:X, Y:]} f = F\text{-}\sum_{Y} g$.

(35) For all matrices A, B, C over K such that $\mathrm{width}\, A = \mathrm{len}\, B$ and $\mathrm{width}\, B = \mathrm{len}\, C$ holds $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

In the sequel p will be an element of the permutations of n-element set.

Let us consider n, K, let M be a matrix over K of dimension n, and let p be an element of the permutations of n-element set. The functor $p\text{-Path}\, M$ yields a finite sequence of elements of the carrier of K and is defined as follows:

(Def.7) $\mathrm{len}(p\text{-Path}\, M) = n$ and for all i, j such that $i \in \mathrm{dom}(p\text{-Path}\, M)$ and $j = p(i)$ holds $(p\text{-Path}\, M)(i) = M_{i,j}$.

Let us consider n, K and let M be a matrix over K of dimension n. The product on paths of M yields a function from the permutations of n-element set into the carrier of K and is defined by the condition (Def.8).

(Def.8) Let p be an element of the permutations of n-element set. Then (the product on paths of M)(p) = $(-1)^{\mathrm{sgn}(p)}$(the multiplication of $K \circledast (p\text{-Path}\, M)$).

Let us consider n, let us consider K, and let M be a matrix over K of dimension n. The functor $\mathrm{Det}\, M$ yields an element of the carrier of K and is defined as follows:

(Def.9) $\mathrm{Det}\, M$ = (the addition of K)$\text{-}\sum_{\Omega^{\mathrm{f}}_{\text{the permutations of } n\text{-element set}}}$(the product on paths of M).

In the sequel a will be an element of the carrier of K.

The following proposition is true

(36) $\mathrm{Det}\langle\langle a\rangle\rangle = a$.

Let us consider n, let us consider K, and let M be a matrix over K of dimension n. The diagonal of M yields a finite sequence of elements of the carrier of K and is defined as follows:

(Def.10) len (the diagonal of M) $= n$ and for every i such that $i \in \mathrm{Seg}\, n$ holds (the diagonal of M)$(i) = M_{i,i}$.

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