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On Defining Functions on Binary Trees [1]

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Summary. This article is a continuation of an article on defining functions on trees (see [6]). In this article we develop terminology specialized for binary trees, first defining binary trees and binary grammars. We recast the induction principle for the set of parse trees of binary grammars and the scheme of defining functions inductively with the set as domain. We conclude with defining the scheme of defining such functions by lambda-like expressions.

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The terminology and notation used here are introduced in the following articles: [12], [14], [15], [13], [8], [9], [5], [7], [11], [10], [1], [3], [4], [2], and [6].

Let D be a non empty set and let t be a tree decorated with elements of D. The root label of t is an element of D and is defined by:

(Def.1) The root label of $t = t(\varepsilon)$.

One can prove the following two propositions:

(1) Let D be a non empty set and let t be a tree decorated with elements of D. Then the roots of $\langle t \rangle = \langle$the root label of $t\rangle$.

(2) Let D be a non empty set and let t_1, t_2 be trees decorated with elements of D. Then the roots of $\langle t_1, t_2 \rangle = \langle$the root label of t_1, the root label of $t_2\rangle$.

A tree is binary if:

(Def.2) For every element t of it such that $t \notin$ Leaves(it) holds $\operatorname{succ} t = \{t \mathbin{^\frown} \langle 0 \rangle, t \mathbin{^\frown} \langle 1 \rangle\}$.

The following propositions are true:

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(3) For every tree T and for every element t of T holds $t \in \mathrm{Leaves}(T)$ iff $t \mathbin{^\frown} \langle 0 \rangle \notin T$.

(4) For every tree T and for every element t of T holds $t \in \mathrm{Leaves}(T)$ iff it is not true that there exists a natural number n such that $t \mathbin{^\frown} \langle n \rangle \in T$.

(5) For every tree T and for every element t of T holds $\mathrm{succ}\, t = \emptyset$ iff $t \in \mathrm{Leaves}(T)$.

(6) The elementary tree of 0 is binary.

(7) The elementary tree of 2 is binary.

Let us note that there exists a tree which is binary and finite.

A decorated tree is binary if:

(Def.3) dom it is binary.

Let D be a non empty set. Observe that there exists a tree decorated with elements of D which is binary and finite.

Let us mention that there exists a decorated tree which is binary and finite.

Let us observe that every tree which is binary is also finite-order.

We now state four propositions:

(8) Let T_0, T_1 be trees and let t be an element of $\overbrace{T_0, T_1}$. Then

(i) for every element p of T_0 such that $t = \langle 0 \rangle \mathbin{^\frown} p$ holds $t \in \mathrm{Leaves}(\overbrace{T_0, T_1})$ iff $p \in \mathrm{Leaves}(T_0)$, and

(ii) for every element p of T_1 such that $t = \langle 1 \rangle \mathbin{^\frown} p$ holds $t \in \mathrm{Leaves}(\overbrace{T_0, T_1})$ iff $p \in \mathrm{Leaves}(T_1)$.

(9) Let T_0, T_1 be trees and let t be an element of $\overbrace{T_0, T_1}$. Then

(i) if $t = \varepsilon$, then $\mathrm{succ}\, t = \{t \mathbin{^\frown} \langle 0 \rangle, t \mathbin{^\frown} \langle 1 \rangle\}$,

(ii) for every element p of T_0 such that $t = \langle 0 \rangle \mathbin{^\frown} p$ and for every finite sequence s_1 holds $s_1 \in \mathrm{succ}\, p$ iff $\langle 0 \rangle \mathbin{^\frown} s_1 \in \mathrm{succ}\, t$, and

(iii) for every element p of T_1 such that $t = \langle 1 \rangle \mathbin{^\frown} p$ and for every finite sequence s_1 holds $s_1 \in \mathrm{succ}\, p$ iff $\langle 1 \rangle \mathbin{^\frown} s_1 \in \mathrm{succ}\, t$.

(10) For all trees T_1, T_2 holds T_1 is binary and T_2 is binary iff $\overbrace{T_1, T_2}$ is binary.

(11) For all decorated trees T_1, T_2 and for arbitrary x holds T_1 is binary and T_2 is binary iff x-tree(T_1, T_2) is binary.

Let D be a non empty set, let x be an element of D, and let T_1, T_2 be binary finite trees decorated with elements of D. Then x-tree(T_1, T_2) is a binary finite tree decorated with elements of D.

A non empty tree construction structure is binary if:

(Def.4) For every symbol s of it and for every finite sequence p such that $s \Rightarrow p$ there exist symbols x_1, x_2 of it such that $p = \langle x_1, x_2 \rangle$.

One can check that there exists a non empty tree construction structure which is binary and strict and has terminals, nonterminals, and useful nonterminals.

The scheme *BinDTConstrStrEx* concerns a non empty set $\mathcal{A}$ and a ternary predicate $\mathcal{P}$, and states that:

There exists a binary strict non empty tree construction structure G such that the carrier of $G = \mathcal{A}$ and for all symbols x, y, z of G holds $x \Rightarrow \langle y, z \rangle$ iff $\mathcal{P}[x, y, z]$

for all values of the parameters.

One can prove the following proposition

(12) Let G be a binary non empty tree construction structure with terminals and nonterminals, and let t_3 be a finite sequence of elements of $\mathrm{TS}(G)$, and let n_1 be a symbol of G. Suppose $n_1 \Rightarrow$ the roots of t_3. Then

(i) n_1 is a nonterminal of G,

(ii) $\mathrm{dom}\, t_3 = \{1, 2\}$,

(iii) $1 \in \mathrm{dom}\, t_3$,

(iv) $2 \in \mathrm{dom}\, t_3$, and

(v) there exist elements t_4, t_5 of $\mathrm{TS}(G)$ such that the roots of $t_3 = \langle$the root label of t_4, the root label of $t_5\rangle$ and $t_4 = t_3(1)$ and $t_5 = t_3(2)$ and n_1-tree$(t_3) = n_1$-tree(t_4, t_5) and $t_4 \in \mathrm{rng}\, t_3$ and $t_5 \in \mathrm{rng}\, t_3$.

Now we present three schemes. The scheme *BinDTConstrInd* concerns a binary non empty tree construction structure $\mathcal{A}$ with terminals and nonterminals and a unary predicate $\mathcal{P}$, and states that:

For every element t of $\mathrm{TS}(\mathcal{A})$ holds $\mathcal{P}[t]$

provided the parameters have the following properties:

- For every terminal s of $\mathcal{A}$ holds $\mathcal{P}[$the root tree of $s]$,
- Let n_1 be a nonterminal of $\mathcal{A}$ and let t_4, t_5 be elements of $\mathrm{TS}(\mathcal{A})$. Suppose $n_1 \Rightarrow \langle$the root label of t_4, the root label of $t_5\rangle$ and $\mathcal{P}[t_4]$ and $\mathcal{P}[t_5]$. Then $\mathcal{P}[n_1$-tree$(t_4, t_5)]$.

The scheme *BinDTConstrIndDef* concerns a binary non empty tree construction structure $\mathcal{A}$ with terminals, nonterminals, and useful nonterminals, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a 5-ary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and states that:

There exists a function f from $\mathrm{TS}(\mathcal{A})$ into $\mathcal{B}$ such that

(i) for every terminal t of $\mathcal{A}$ holds $f($the root tree of $t) = \mathcal{F}(t)$, and

(ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_4, t_5 of $\mathrm{TS}(\mathcal{A})$ and for all symbols r_1, r_2 of $\mathcal{A}$ such that $r_1 =$ the root label of t_4 and $r_2 =$ the root label of t_5 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ and for all elements x_3, x_4 of $\mathcal{B}$ such that $x_3 = f(t_4)$ and $x_4 = f(t_5)$ holds $f(n_1\text{-tree}(t_4, t_5)) = \mathcal{G}(n_1, r_1, r_2, x_3, x_4)$

for all values of the parameters.

The scheme *BinDTConstrUniqDef* deals with a binary non empty tree construction structure $\mathcal{A}$ with terminals, nonterminals, and useful nonterminals, a non empty set $\mathcal{B}$, functions $\mathcal{C}$, $\mathcal{D}$ from $\mathrm{TS}(\mathcal{A})$ into $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a 5-ary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and states that:

$\mathcal{C} = \mathcal{D}$

provided the following requirements are met:

- (i) For every terminal t of $\mathcal{A}$ holds $\mathcal{C}$(the root tree of t) $= \mathcal{F}(t)$, and
  (ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_4, t_5 of TS($\mathcal{A}$) and for all symbols r_1, r_2 of $\mathcal{A}$ such that r_1 = the root label of t_4 and r_2 = the root label of t_5 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ and for all elements x_3, x_4 of $\mathcal{B}$ such that $x_3 = \mathcal{C}(t_4)$ and $x_4 = \mathcal{C}(t_5)$ holds $\mathcal{C}(n_1\text{-tree}(t_4, t_5)) = \mathcal{G}(n_1, r_1, r_2, x_3, x_4)$,
- (i) For every terminal t of $\mathcal{A}$ holds $\mathcal{D}$(the root tree of t) $= \mathcal{F}(t)$, and
  (ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_4, t_5 of TS($\mathcal{A}$) and for all symbols r_1, r_2 of $\mathcal{A}$ such that r_1 = the root label of t_4 and r_2 = the root label of t_5 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ and for all elements x_3, x_4 of $\mathcal{B}$ such that $x_3 = \mathcal{D}(t_4)$ and $x_4 = \mathcal{D}(t_5)$ holds $\mathcal{D}(n_1\text{-tree}(t_4, t_5)) = \mathcal{G}(n_1, r_1, r_2, x_3, x_4)$.

Let A, B, C be non empty sets, let a be an element of A, let b be an element of B, and let c be an element of C. Then $\langle a, b, c \rangle$ is an element of $[: A, B, C :]$.

Now we present two schemes. The scheme *BinDTC DefLambda* deals with a binary non empty tree construction structure $\mathcal{A}$ with terminals, nonterminals, and useful nonterminals, non empty sets $\mathcal{B}$, $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a 4-ary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

> There exists a function f from TS($\mathcal{A}$) into $\mathcal{C}^{\mathcal{B}}$ such that
> (i) for every terminal t of $\mathcal{A}$ there exists a function g from $\mathcal{B}$ into $\mathcal{C}$ such that $g = f$(the root tree of t) and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{F}(t, a)$, and
> (ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_1, t_2 of TS($\mathcal{A}$) and for all symbols r_1, r_2 of $\mathcal{A}$ such that r_1 = the root label of t_1 and r_2 = the root label of t_2 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ there exist functions g, f_1, f_2 from $\mathcal{B}$ into $\mathcal{C}$ such that $g = f(n_1\text{-tree}(t_1, t_2))$ and $f_1 = f(t_1)$ and $f_2 = f(t_2)$ and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{G}(n_1, f_1, f_2, a)$

for all values of the parameters.

The scheme *BinDTC DefLambdaUniq* deals with a binary non empty tree construction structure $\mathcal{A}$ with terminals, nonterminals, and useful nonterminals, non empty sets $\mathcal{B}$, $\mathcal{C}$, functions $\mathcal{D}$, $\mathcal{E}$ from TS($\mathcal{A}$) into $\mathcal{C}^{\mathcal{B}}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a 4-ary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

> $\mathcal{D} = \mathcal{E}$

provided the parameters satisfy the following conditions:

- (i) For every terminal t of $\mathcal{A}$ there exists a function g from $\mathcal{B}$ into $\mathcal{C}$ such that $g = \mathcal{D}$(the root tree of t) and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{F}(t, a)$, and
  (ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_1, t_2 of TS($\mathcal{A}$) and for all symbols r_1, r_2 of $\mathcal{A}$ such that r_1 = the root label of t_1 and r_2 = the root label of t_2 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ there exist functions g, f_1, f_2 from $\mathcal{B}$ into $\mathcal{C}$ such that $g = \mathcal{D}(n_1\text{-tree}(t_1, t_2))$

and $f_1 = \mathcal{D}(t_1)$ and $f_2 = \mathcal{D}(t_2)$ and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{G}(n_1, f_1, f_2, a)$,

- (i) For every terminal t of $\mathcal{A}$ there exists a function g from $\mathcal{B}$ into $\mathcal{C}$ such that $g = \mathcal{E}$(the root tree of t) and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{F}(t, a)$, and

  (ii) for every nonterminal n_1 of $\mathcal{A}$ and for all elements t_1, t_2 of TS($\mathcal{A}$) and for all symbols r_1, r_2 of $\mathcal{A}$ such that $r_1 =$ the root label of t_1 and $r_2 =$ the root label of t_2 and $n_1 \Rightarrow \langle r_1, r_2 \rangle$ there exist functions g, f_1, f_2 from $\mathcal{B}$ into $\mathcal{C}$ such that $g = \mathcal{E}(n_1\text{-tree}(t_1, t_2))$ and $f_1 = \mathcal{E}(t_1)$ and $f_2 = \mathcal{E}(t_2)$ and for every element a of $\mathcal{B}$ holds $g(a) = \mathcal{G}(n_1, f_1, f_2, a)$.

Let G be a binary non empty tree construction structure with terminals and nonterminals. Note that every element of TS(G) is binary.

References

[1] Grzegorz Bancerek. Introduction to trees. *Formalized Mathematics*, 1(**2**):421–427, 1990.
[2] Grzegorz Bancerek. Joining of decorated trees. *Formalized Mathematics*, 4(**1**):77–82, 1993.
[3] Grzegorz Bancerek. König's lemma. *Formalized Mathematics*, 2(**3**):397–402, 1991.
[4] Grzegorz Bancerek. Sets and functions of trees and joining operations of trees. *Formalized Mathematics*, 3(**2**):195–204, 1992.
[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[6] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on trees. *Formalized Mathematics*, 4(**1**):91–101, 1993.
[7] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[10] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar - part 1. *Formalized Mathematics*, 2(**5**):683–687, 1991.
[11] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[13] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[14] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[15] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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