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Some Properties of the Intervals

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The papers [8], [10], [4], [5], [6], [1], [2], [3], [7], and [9] provide the terminology and notation for this paper.

The scheme *FunctXD YD* concerns a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a function F from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $\mathcal{P}[x, F(x)]$

provided the following condition is satisfied:

- For every element x of $\mathcal{A}$ there exists an element y of $\mathcal{B}$ such that $\mathcal{P}[x, y]$.

Let X, Y be non empty sets. Note that Y^X is non empty.

We now state a number of propositions:

(1) There exists a function F from $\mathbb{N}$ into $[\!: \mathbb{N}, \mathbb{N} :\!]$ such that F is one-to-one and $\operatorname{dom} F = \mathbb{N}$ and $\operatorname{rng} F = [\!: \mathbb{N}, \mathbb{N} :\!]$.

(2) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative holds $0_{\overline{\mathbb{R}}} \leq \sum F$.

(3) Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and let x be a *Real number*. Suppose there exists a natural number n such that $x \leq F(n)$ and F is non-negative. Then $x \leq \sum F$.

(4) For every *Real number* x such that there exists a *Real number* y such that $y < x$ holds $x \neq -\infty$.

(5) For every *Real number* x such that there exists a *Real number* y such that $x < y$ holds $x \neq +\infty$.

(6) For all *Real numbers* x, y holds $x \leq y$ iff $x < y$ or $x = y$.

(7) Let x, y be *Real numbers* and let p, q be real numbers. If $x = p$ and $y = q$, then $p \leq q$ iff $x \leq y$.

(8) For all *Real numbers* x, y such that x is a real number holds $(y-x)+x = y$ and $(y + x) - x = y$.

(9) For all *Real numbers* x, y such that $x \in \mathbb{R}$ holds $x + y = y + x$.

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(10) For all *Real numbers* x, y, z such that $z \in \mathbb{R}$ and $y < x$ holds $(z + x) - (z + y) = x - y$.

(11) For all *Real numbers* x, y, z such that $z \in \mathbb{R}$ and $x \leq y$ holds $z+x \leq z+y$ and $x + z \leq y + z$ and $x - z \leq y - z$.

(12) For all *Real numbers* x, y, z such that $z \in \mathbb{R}$ and $x < y$ holds $z+x < z+y$ and $x + z < y + z$ and $x - z < y - z$.

Let x be a real number. The functor $\overline{\mathbb{R}}(x)$ yields a *Real number* and is defined as follows:

(Def.1) $\overline{\mathbb{R}}(x) = x$.

The following propositions are true:

(13) For all real numbers x, y holds $x \leq y$ iff $\overline{\mathbb{R}}(x) \leq \overline{\mathbb{R}}(y)$.

(14) For all real numbers x, y holds $x < y$ iff $\overline{\mathbb{R}}(x) < \overline{\mathbb{R}}(y)$.

(15) For all *Real numbers* x, y, z such that $x < y$ and $y < z$ holds y is a real number.

(16) Let x, y, z be *Real numbers*. Suppose x is a real number and z is a real number and $x \leq y$ and $y \leq z$. Then y is a real number.

(17) For all *Real numbers* x, y, z such that x is a real number and $x \leq y$ and $y < z$ holds y is a real number.

(18) For all *Real numbers* x, y, z such that $x < y$ and $y \leq z$ and z is a real number holds y is a real number.

(19) For all *Real numbers* x, y such that $0_{\overline{\mathbb{R}}} < x$ and $x < y$ holds $0_{\overline{\mathbb{R}}} < y - x$.

(20) For all *Real numbers* x, y, z such that $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq z$ and $z + x < y$ holds $z < y - x$.

(21) For every *Real number* x holds $x - 0_{\overline{\mathbb{R}}} = x$.

(22) For all *Real numbers* x, y, z such that $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq z$ and $z + x < y$ holds $z \leq y$.

(23) For every *Real number* x such that $0_{\overline{\mathbb{R}}} < x$ there exists a *Real number* y such that $0_{\overline{\mathbb{R}}} < y$ and $y < x$.

(24) Let x, z be *Real numbers*. Suppose $0_{\overline{\mathbb{R}}} < x$ and $x < z$. Then there exists a *Real number* y such that $0_{\overline{\mathbb{R}}} < y$ and $x + y < z$ and $y \in \mathbb{R}$.

(25) Let x, z be *Real numbers*. Suppose $0_{\overline{\mathbb{R}}} \leq x$ and $x < z$. Then there exists a *Real number* y such that $0_{\overline{\mathbb{R}}} < y$ and $x + y < z$ and $y \in \mathbb{R}$.

(26) For every *Real number* x such that $0_{\overline{\mathbb{R}}} < x$ there exists a *Real number* y such that $0_{\overline{\mathbb{R}}} < y$ and $y + y < x$.

Let x be a *Real number*. Let us assume that $0_{\overline{\mathbb{R}}} < x$. The functor $\operatorname{Seg} x$ yields a non empty subset of $\overline{\mathbb{R}}$ and is defined by:

(Def.2) For every *Real number* y holds $y \in \operatorname{Seg} x$ iff $0_{\overline{\mathbb{R}}} < y$ and $y + y < x$.

Let x be a *Real number*. Let us assume that $0_{\overline{\mathbb{R}}} < x$. The functor $\operatorname{len} x$ yielding a *Real number* is defined as follows:

(Def.3) $\operatorname{len} x = \sup \operatorname{Seg} x$.

Next we state several propositions:

(27) For every *Real number* x such that $0_{\overline{\mathbb{R}}} < x$ holds $0_{\overline{\mathbb{R}}} < \operatorname{len} x$.

(28) For every *Real number* x such that $0_{\overline{\mathbb{R}}} < x$ holds $\operatorname{len} x \leq x$.

(29) For every *Real number* x such that $0_{\overline{\mathbb{R}}} < x$ and $x < +\infty$ holds $\operatorname{len} x$ is a real number.

(30) For every *Real number* x such that $0_{\overline{\mathbb{R}}} < x$ holds $\operatorname{len} x + \operatorname{len} x \leq x$.

(31) Let e_1 be a *Real number*. Suppose $0_{\overline{\mathbb{R}}} < e_1$. Then there exists a function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every natural number n holds $0_{\overline{\mathbb{R}}} < F(n)$ and $\sum F < e_1$.

(32) Let e_1 be a *Real number* and let X be a non empty subset of $\overline{\mathbb{R}}$. Suppose $0_{\overline{\mathbb{R}}} < e_1$ and $\inf X$ is a real number. Then there exists a *Real number* x such that $x \in X$ and $x < \inf X + e_1$.

(33) Let e_1 be a *Real number* and let X be a non empty subset of $\overline{\mathbb{R}}$. Suppose $0_{\overline{\mathbb{R}}} < e_1$ and $\sup X$ is a real number. Then there exists a *Real number* x such that $x \in X$ and $\sup X - e_1 < x$.

(34) Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose F is non-negative and $\sum F < +\infty$. Let n be a natural number. Then $F(n) \in \mathbb{R}$.

$-\infty$ is a *Real number*.

$+\infty$ is a *Real number*.

We now state a number of propositions:

(35) $\mathbb{R}$ is an interval and $\mathbb{R} =]-\infty, +\infty[$ and $\mathbb{R} = [-\infty, +\infty]$ and $\mathbb{R} = [-\infty, +\infty[$ and $\mathbb{R} =]-\infty, +\infty]$.

(36) For all *Real numbers* a, b such that $b = -\infty$ holds $]a, b[= \emptyset$ and $[a, b] = \emptyset$ and $[a, b[= \emptyset$ and $]a, b] = \emptyset$.

(37) For all *Real numbers* a, b such that $a = +\infty$ holds $]a, b[= \emptyset$ and $[a, b] = \emptyset$ and $[a, b[= \emptyset$ and $]a, b] = \emptyset$.

(38) Let A be an interval and let a, b be *Real numbers*. Suppose $A =]a, b[$. Let c, d be real numbers. Suppose $c \in A$ and $d \in A$. Let e be a real number. If $c \leq e$ and $e \leq d$, then $e \in A$.

(39) Let A be an interval and let a, b be *Real numbers*. Suppose $A = [a, b]$. Let c, d be real numbers. Suppose $c \in A$ and $d \in A$. Let e be a real number. If $c \leq e$ and $e \leq d$, then $e \in A$.

(40) Let A be an interval and let a, b be *Real numbers*. Suppose $A =]a, b]$. Let c, d be real numbers. Suppose $c \in A$ and $d \in A$. Let e be a real number. If $c \leq e$ and $e \leq d$, then $e \in A$.

(41) Let A be an interval and let a, b be *Real numbers*. Suppose $A = [a, b[$. Let c, d be real numbers. Suppose $c \in A$ and $d \in A$. Let e be a real number. If $c \leq e$ and $e \leq d$, then $e \in A$.

(42) Let A be a non empty subset of $\overline{\mathbb{R}}$ and let m, M be *Real numbers*. Suppose $m = \inf A$ and $M = \sup A$. Suppose that

(i) for all real numbers c, d such that $c \in A$ and $d \in A$ and for every real number e such that $c \leq e$ and $e \leq d$ holds $e \in A$,

(ii) $m \notin A$, and

(iii) $M \notin A$.

Then $A =]m, M[$.

(43) Let A be a non empty subset of $\overline{\mathbb{R}}$ and let m, M be *Real numbers*. Suppose $m = \inf A$ and $M = \sup A$. Suppose that

(i) for all real numbers c, d such that $c \in A$ and $d \in A$ and for every real number e such that $c \le e$ and $e \le d$ holds $e \in A$,

(ii) $m \in A$,

(iii) $M \in A$, and

(iv) $A \subseteq \mathbb{R}$.

Then $A = [m, M]$.

(44) Let A be a non empty subset of $\overline{\mathbb{R}}$ and let m, M be *Real numbers*. Suppose $m = \inf A$ and $M = \sup A$. Suppose that

(i) for all real numbers c, d such that $c \in A$ and $d \in A$ and for every real number e such that $c \le e$ and $e \le d$ holds $e \in A$,

(ii) $m \in A$,

(iii) $M \notin A$, and

(iv) $A \subseteq \mathbb{R}$.

Then $A = [m, M[$.

(45) Let A be a non empty subset of $\overline{\mathbb{R}}$ and let m, M be *Real numbers*. Suppose $m = \inf A$ and $M = \sup A$. Suppose that

(i) for all real numbers c, d such that $c \in A$ and $d \in A$ and for every real number e such that $c \le e$ and $e \le d$ holds $e \in A$,

(ii) $m \notin A$,

(iii) $M \in A$, and

(iv) $A \subseteq \mathbb{R}$.

Then $A =]m, M]$.

(46) Let A be a subset of $\mathbb{R}$. Then A is an interval if and only if for all real numbers a, b such that $a \in A$ and $b \in A$ and for every real number c such that $a \le c$ and $c \le b$ holds $c \in A$.

Let A, B be intervals. Then $A \cup B$ is a subset of $\mathbb{R}$.

Next we state the proposition

(47) For all intervals A, B such that $A \cap B \neq \emptyset$ holds $A \cup B$ is an interval.

Let A be an interval. Let us assume that $A \neq \emptyset$. The functor $\inf A$ yields a *Real number* and is defined as follows:

(Def.4) There exists a *Real number* b such that $\inf A \le b$ but $A =]\inf A, b[$ or $A =]\inf A, b]$ or $A = [\inf A, b]$ or $A = [\inf A, b[$.

Let A be an interval. Let us assume that $A \neq \emptyset$. The functor $\sup A$ yielding a *Real number* is defined as follows:

(Def.5) There exists a *Real number* a such that $a \le \sup A$ but $A =]a, \sup A[$ or $A =]a, \sup A]$ or $A = [a, \sup A]$ or $A = [a, \sup A[$.

Next we state a number of propositions:

(48) For every interval A such that A is open interval and $A \neq \emptyset$ holds $\inf A \le \sup A$ and $A =]\inf A, \sup A[$.

(49) For every interval A such that A is closed interval and $A \neq \emptyset$ holds $\inf A \leq \sup A$ and $A = [\inf A, \sup A]$.

(50) For every interval A such that A is right open interval and $A \neq \emptyset$ holds $\inf A \leq \sup A$ and $A = [\inf A, \sup A[$.

(51) For every interval A such that A is left open interval and $A \neq \emptyset$ holds $\inf A \leq \sup A$ and $A =]\inf A, \sup A]$.

(52) For every interval A such that $A \neq \emptyset$ holds $\inf A \leq \sup A$ but $A =]\inf A, \sup A[$ or $A =]\inf A, \sup A]$ or $A = [\inf A, \sup A]$ or $A = [\inf A, \sup A[$.

(53) For all intervals A, B such that $A = \emptyset$ or $B = \emptyset$ holds $A \cup B$ is an interval.

(54) For every interval A and for every real number a such that $a \in A$ holds $\inf A \leq \overline{\mathbb{R}}(a)$ and $\overline{\mathbb{R}}(a) \leq \sup A$.

(55) For all intervals A, B and for all real numbers a, b such that $a \in A$ and $b \in B$ holds if $\sup A \leq \inf B$, then $a \leq b$.

(56) For every interval A and for every *Real number* a such that $a \in A$ holds $\inf A \leq a$ and $a \leq \sup A$.

(57) For every interval A such that $A \neq \emptyset$ and for every *Real number* a such that $\inf A < a$ and $a < \sup A$ holds $a \in A$.

(58) For all intervals A, B such that $\sup A = \inf B$ but $\sup A \in A$ or $\inf B \in B$ holds $A \cup B$ is an interval.

Let A be a subset of $\mathbb{R}$ and let x be a real number. The functor $x + A$ yields a subset of $\mathbb{R}$ and is defined by:

(Def.6) For every real number y holds $y \in x + A$ iff there exists a real number z such that $z \in A$ and $y = x + z$.

One can prove the following propositions:

(59) For every subset A of $\mathbb{R}$ and for every real number x holds $-x+(x+A) = A$.

(60) For every real number x and for every subset A of $\mathbb{R}$ such that $A = \mathbb{R}$ holds $x + A = A$.

(61) For every real number x holds $x + \emptyset = \emptyset$.

(62) For every interval A and for every real number x holds A is open interval iff $x + A$ is open interval.

(63) For every interval A and for every real number x holds A is closed interval iff $x + A$ is closed interval.

(64) Let A be an interval and let x be a real number. Then A is right open interval if and only if $x + A$ is right open interval.

(65) Let A be an interval and let x be a real number. Then A is left open interval if and only if $x + A$ is left open interval.

(66) For every interval A and for every real number x holds $x + A$ is an interval.

Let A be an interval and let x be a real number. Note that $x+A$ is interval. The following proposition is true

(67) For every interval A and for every real number x holds $\mathrm{vol}(A) = \mathrm{vol}(x+A)$.

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