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# Subalgebras of Many Sorted Algebra. Lattice of Subalgebras

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The articles [12], [13], [5], [6], [2], [8], [9], [7], [4], [14], [3], [1], [11], and [10] provide the notation and terminology for this paper.

## 1. Auxilary Facts about Many Sorted Sets

In this paper $x$ will be arbitrary.

The scheme *LambdaB* concerns a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

> There exists a function $f$ such that $\mathrm{dom}\, f = \mathcal{A}$ and for every element $d$ of $\mathcal{A}$ holds $f(d) = \mathcal{F}(d)$

for all values of the parameters.

Let $I$ be a set, let $X$ be a many sorted set of $I$, and let $Y$ be a non-empty many sorted set of $I$. Observe that $X \cup Y$ is non-empty and $Y \cup X$ is non-empty.

Next we state two propositions:

(1) Let $I$ be a set, and let $X$ be a many sorted set of $I$, and let $Y$ be a non-empty many sorted set of $I$. Then $X \cup Y$ is non-empty and $Y \cup X$ is non-empty.

(2) For every non empty set $I$ and for all many sorted sets $X$, $Y$ of $I$ and for every element $i$ of $I^*$ holds $\prod((X \cap Y) \cdot i) = \prod(X \cdot i) \cap \prod(Y \cdot i)$.

Let $I$ be a set and let $M$ be a many sorted set of $I$. A many sorted set of $I$ is said to be a many sorted subset of $M$ if:

(Def.1) It $\subseteq M$.

Let $I$ be a set and let $M$ be a non-empty many sorted set of $I$. Observe that there exists a many sorted subset of $M$ which is non-empty.




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## 2. Constants of a Many Sorted Algebra

We follow the rules: $S$ will denote a non void non empty many sorted signature, $o$ will denote an operation symbol of $S$, and $U_0$, $U_1$, $U_2$ will denote algebras over $S$.

Let $S$ be a non empty many sorted signature and let $U_0$ be an algebra over $S$. A subset of $U_0$ is a many sorted subset of the sorts of $U_0$.

Let $S$ be a non empty many sorted signature. A sort symbol of $S$ has constants if:

(Def.2) There exists an operation symbol $o$ of $S$ such that (the arity of $S$)$(o) = \varepsilon$ and (the result sort of $S$)$(o) =$ it.

A non empty many sorted signature has constant operations if:

(Def.3) Every sort symbol of it has constants.

Let $A$ be a non empty set, let $B$ be a set, let $a$ be a function from $B$ into $A^*$, and let $r$ be a function from $B$ into $A$. Note that $\langle A, B, a, r \rangle$ is non empty.

Let us observe that there exists a non empty many sorted signature which is non void and strict and has constant operations.

Let $S$ be a non void non empty many sorted signature, let $U_0$ be an algebra over $S$, and let $s$ be a sort symbol of $S$. The functor Constants$(U_0, s)$ yielding a subset of (the sorts of $U_0$)$(s)$ is defined by:

(Def.4) (i) There exists a non empty set $A$ such that $A =$ (the sorts of $U_0$)$(s)$ and Constants$(U_0, s) = \{a : a$ ranges over elements of $A$, $\bigvee_o$ (the arity of $S$)$(o) = \varepsilon \ \wedge$ (the result sort of $S$)$(o) = s \ \wedge \ a \in \operatorname{rng} \operatorname{Den}(o, U_0)\}$ if (the sorts of $U_0$)$(s) \neq \emptyset$,

(ii) Constants$(U_0, s) = \emptyset$, otherwise.

Let $S$ be a non void non empty many sorted signature and let $U_0$ be an algebra over $S$. The functor Constants$(U_0)$ yielding a subset of $U_0$ is defined as follows:

(Def.5) For every sort symbol $s$ of $S$ holds (Constants$(U_0)$)$(s) =$ Constants$(U_0, s)$.

Let $S$ be a non void non empty many sorted signature with constant operations, let $U_0$ be a non-empty algebra over $S$, and let $s$ be a sort symbol of $S$. One can verify that Constants$(U_0, s)$ is non empty.

Let $S$ be a non void non empty many sorted signature with constant operations and let $U_0$ be a non-empty algebra over $S$. One can verify that Constants$(U_0)$ is non-empty.

## 3. Subalgebras of a Many Sorted Algebra

Let $S$ be a non void non empty many sorted signature, let $U_0$ be an algebra over $S$, let $o$ be an operation symbol of $S$, and let $A$ be a subset of $U_0$. We say that $A$ is closed on $o$ if and only if:

(Def.6) $\mathrm{rng}(\mathrm{Den}(o, U_0) \restriction (A^{\#} \cdot (\text{the arity of } S))(o)) \subseteq (A \cdot (\text{the result sort of } S))(o)$.

Let $S$ be a non void non empty many sorted signature, let $U_0$ be an algebra over $S$, and let $A$ be a subset of $U_0$. We say that $A$ is operations closed if and only if:

(Def.7) For every operation symbol $o$ of $S$ holds $A$ is closed on $o$.

One can prove the following proposition

(3) Let $S$ be a non void non empty many sorted signature, and let $o$ be an operation symbol of $S$, and let $U_0$ be an algebra over $S$, and let $B_0$, $B_1$ be subsets of $U_0$. If $B_0 \subseteq B_1$, then $({B_0}^{\#} \cdot (\text{the arity of } S))(o) \subseteq ({B_1}^{\#} \cdot (\text{the arity of } S))(o)$.

Let $S$ be a non void non empty many sorted signature, let $U_0$ be an algebra over $S$, let $o$ be an operation symbol of $S$, and let $A$ be a subset of $U_0$. Let us assume that $A$ is closed on $o$. The functor $o_A$ yielding a function from $(A^{\#} \cdot (\text{the arity of } S))(o)$ into $(A \cdot (\text{the result sort of } S))(o)$ is defined as follows:

(Def.8) $o_A = \mathrm{Den}(o, U_0) \restriction (A^{\#} \cdot (\text{the arity of } S))(o)$.

Let $S$ be a non void non empty many sorted signature, let $U_0$ be an algebra over $S$, and let $A$ be a subset of $U_0$. The functor $\mathrm{Opers}(U_0, A)$ yielding a many sorted function from $A^{\#} \cdot (\text{the arity of } S)$ into $A \cdot (\text{the result sort of } S)$ is defined by:

(Def.9) For every operation symbol $o$ of $S$ holds $(\mathrm{Opers}(U_0, A))(o) = o_A$.

Next we state two propositions:

(4) Let $U_0$ be an algebra over $S$ and let $B$ be a subset of $U_0$. Suppose $B$ = the sorts of $U_0$. Then $B$ is operations closed and for every $o$ holds $o_B = \mathrm{Den}(o, U_0)$.

(5) For every subset $B$ of $U_0$ such that $B$ = the sorts of $U_0$ holds $\mathrm{Opers}(U_0, B)$ = the characteristics of $U_0$.

Let $S$ be a non void non empty many sorted signature and let $U_0$ be an algebra over $S$. An algebra over $S$ is called a subalgebra of $U_0$ if it satisfies the conditions (Def.10).

(Def.10) (i) The sorts of it is a subset of $U_0$, and

(ii) for every subset $B$ of $U_0$ such that $B$ = the sorts of it holds $B$ is operations closed and the characteristics of it = $\mathrm{Opers}(U_0, B)$.

Let $S$ be a non void non empty many sorted signature and let $U_0$ be an algebra over $S$. One can check that there exists a subalgebra of $U_0$ which is strict.

Let $S$ be a non void non empty many sorted signature and let $U_0$ be a non-empty algebra over $S$. Observe that there exists a subalgebra of $U_0$ which is non-empty and strict.

One can prove the following propositions:

(6) $U_0$ is a subalgebra of $U_0$.

(7) If $U_0$ is a subalgebra of $U_1$ and $U_1$ is a subalgebra of $U_2$, then $U_0$ is a subalgebra of $U_2$.

(8) If $U_1$ is a strict subalgebra of $U_2$ and $U_2$ is a strict subalgebra of $U_1$, then $U_1 = U_2$.

(9) For all subalgebras $U_1$, $U_2$ of $U_0$ such that the sorts of $U_1 \subseteq$ the sorts of $U_2$ holds $U_1$ is a subalgebra of $U_2$.

(10) For all strict subalgebras $U_1$, $U_2$ of $U_0$ such that the sorts of $U_1 =$ the sorts of $U_2$ holds $U_1 = U_2$.

(11) Let $S$ be a non void non empty many sorted signature, and let $U_0$ be an algebra over $S$, and let $U_1$ be a subalgebra of $U_0$. Then Constants$(U_0)$ is a subset of $U_1$.

(12) Let $S$ be a non void non empty many sorted signature with constant operations, and let $U_0$ be a non-empty algebra over $S$, and let $U_1$ be a non-empty subalgebra of $U_0$. Then Constants$(U_0)$ is a non-empty subset of $U_1$.

(13) Let $S$ be a non void non empty many sorted signature with constant operations, and let $U_0$ be a non-empty algebra over $S$, and let $U_1$, $U_2$ be non-empty subalgebras of $U_0$. Then (the sorts of $U_1$) $\cap$ (the sorts of $U_2$) is non-empty.

## 4. Many Sorted Subsets of Many Sorted Algebra

Let $S$ be a non void non empty many sorted signature, let $U_0$ be an algebra over $S$, and let $A$ be a subset of $U_0$. The functor SubSorts$(A)$ yielding a non empty set is defined by the condition (Def.11).

(Def.11) Let $x$ be arbitrary. Then $x \in$ SubSorts$(A)$ if and only if the following conditions are satisfied:

(i) $x \in (2^{\bigcup (\text{the sorts of } U_0)})^{\text{the carrier of } S}$,

(ii) $x$ is a subset of $U_0$, and

(iii) for every subset $B$ of $U_0$ such that $B = x$ holds $B$ is operations closed and Constants$(U_0) \subseteq B$ and $A \subseteq B$.

Let $S$ be a non void non empty many sorted signature and let $U_0$ be an algebra over $S$. The functor SubSorts$(U_0)$ yields a non empty set and is defined by the condition (Def.12).

(Def.12) Let $x$ be arbitrary. Then $x \in$ SubSorts$(U_0)$ if and only if the following conditions are satisfied:

(i) $x \in (2^{\bigcup (\text{the sorts of } U_0)})^{\text{the carrier of } S}$,

(ii) $x$ is a subset of $U_0$, and

(iii) for every subset $B$ of $U_0$ such that $B = x$ holds $B$ is operations closed.

Let $S$ be a non void non empty many sorted signature, let $U_0$ be an algebra over $S$, and let $e$ be an element of SubSorts$(U_0)$. The functor ${}^{@}e$ yielding a subset of $U_0$ is defined as follows:

(Def.13) $^{@}e = e$.

Next we state two propositions:

(14) For all subsets $A$, $B$ of $U_0$ holds $B \in \mathrm{SubSorts}(A)$ iff $B$ is operations closed and $\mathrm{Constants}(U_0) \subseteq B$ and $A \subseteq B$.

(15) For every subset $B$ of $U_0$ holds $B \in \mathrm{SubSorts}(U_0)$ iff $B$ is operations closed.

Let $S$ be a non void non empty many sorted signature, let $U_0$ be an algebra over $S$, let $A$ be a subset of $U_0$, and let $s$ be a sort symbol of $S$. The functor $\mathrm{SubSort}(A, s)$ yields a non empty set and is defined as follows:

(Def.14) For arbitrary $x$ holds $x \in \mathrm{SubSort}(A, s)$ iff there exists a subset $B$ of $U_0$ such that $B \in \mathrm{SubSorts}(A)$ and $x = B(s)$.

Let $S$ be a non void non empty many sorted signature, let $U_0$ be an algebra over $S$, and let $A$ be a subset of $U_0$. The functor $\mathrm{MSSubSort}(A)$ yields a subset of $U_0$ and is defined as follows:

(Def.15) For every sort symbol $s$ of $S$ holds $(\mathrm{MSSubSort}(A))(s) = \bigcap \mathrm{SubSort}(A, s)$.

We now state several propositions:

(16) For every subset $A$ of $U_0$ holds $\mathrm{Constants}(U_0) \cup A \subseteq \mathrm{MSSubSort}(A)$.

(17) For every subset $A$ of $U_0$ such that $\mathrm{Constants}(U_0) \cup A$ is non-empty holds $\mathrm{MSSubSort}(A)$ is non-empty.

(18) Let $A$ be a subset of $U_0$ and let $B$ be a subset of $U_0$. If $B \in \mathrm{SubSorts}(A)$, then $((\mathrm{MSSubSort}(A))^{\#} \cdot (\text{the arity of } S))(o) \subseteq (B^{\#} \cdot (\text{the arity of } S))(o)$.

(19) Let $A$ be a subset of $U_0$ and let $B$ be a subset of $U_0$. Suppose $B \in \mathrm{SubSorts}(A)$. Then $\mathrm{rng}(\mathrm{Den}(o, U_0) \restriction ((\mathrm{MSSubSort}(A))^{\#} \cdot (\text{the arity of } S))(o)) \subseteq (B \cdot (\text{the result sort of } S))(o)$.

(20) For every subset $A$ of $U_0$ holds $\mathrm{rng}(\mathrm{Den}(o, U_0) \restriction ((\mathrm{MSSubSort}(A))^{\#} \cdot (\text{the arity of } S))(o)) \subseteq (\mathrm{MSSubSort}(A) \cdot (\text{the result sort of } S))(o)$.

(21) For every subset $A$ of $U_0$ holds $\mathrm{MSSubSort}(A)$ is operations closed and $A \subseteq \mathrm{MSSubSort}(A)$.

## 5. Operations on Many Sorted Algebra and its Subalgebras

Let $S$ be a non void non empty many sorted signature, let $U_0$ be an algebra over $S$, and let $A$ be a subset of $U_0$. Let us assume that $A$ is operations closed. The functor $U_0 \restriction A$ yields a strict subalgebra of $U_0$ and is defined as follows:

(Def.16) $U_0 \restriction A = \langle A, (\mathrm{Opers}(U_0, A)$ **qua** many sorted function from $A^{\#} \cdot (\text{the arity of } S)$ into $A \cdot (\text{the result sort of } S))\rangle$.

Let $S$ be a non void non empty many sorted signature, let $U_0$ be an algebra over $S$, and let $U_1$, $U_2$ be subalgebras of $U_0$. The functor $U_1 \cap U_2$ yielding a strict subalgebra of $U_0$ is defined by the conditions (Def.17).

(Def.17) (i) The sorts of $U_1 \cap U_2 =$ (the sorts of $U_1$) $\cap$ (the sorts of $U_2$), and

(ii) for every subset $B$ of $U_0$ such that $B =$ the sorts of $U_1 \cap U_2$ holds $B$ is operations closed and the characteristics of $U_1 \cap U_2 = \mathrm{Opers}(U_0, B)$.

Let $S$ be a non void non empty many sorted signature, let $U_0$ be an algebra over $S$, and let $A$ be a subset of $U_0$. The functor $\mathrm{Gen}(A)$ yields a strict subalgebra of $U_0$ and is defined by the conditions (Def.18).

(Def.18) (i) $A$ is a subset of $\mathrm{Gen}(A)$, and

(ii) for every subalgebra $U_1$ of $U_0$ such that $A$ is a subset of $U_1$ holds $\mathrm{Gen}(A)$ is a subalgebra of $U_1$.

Let $S$ be a non void non empty many sorted signature, let $U_0$ be a non-empty algebra over $S$, and let $A$ be a non-empty subset of $U_0$. Observe that $\mathrm{Gen}(A)$ is non-empty.

We now state three propositions:

(22) Let $S$ be a non void non empty many sorted signature, and let $U_0$ be a strict algebra over $S$, and let $B$ be a subset of $U_0$. If $B =$ the sorts of $U_0$, then $\mathrm{Gen}(B) = U_0$.

(23) Let $S$ be a non void non empty many sorted signature, and let $U_0$ be an algebra over $S$, and let $U_1$ be a strict subalgebra of $U_0$, and let $B$ be a subset of $U_0$. If $B =$ the sorts of $U_1$, then $\mathrm{Gen}(B) = U_1$.

(24) Let $S$ be a non void non empty many sorted signature with constant operations, and let $U_0$ be a non-empty algebra over $S$, and let $U_1$ be a subalgebra of $U_0$. Then $\mathrm{Gen}(\mathrm{Constants}(U_0)) \cap U_1 = \mathrm{Gen}(\mathrm{Constants}(U_0))$.

Let $S$ be a non void non empty many sorted signature, let $U_0$ be a non-empty algebra over $S$, and let $U_1$, $U_2$ be subalgebras of $U_0$. The functor $U_1 \bigsqcup U_2$ yielding a strict subalgebra of $U_0$ is defined as follows:

(Def.19) For every subset $A$ of $U_0$ such that $A =$ (the sorts of $U_1$) $\cup$ (the sorts of $U_2$) holds $U_1 \bigsqcup U_2 = \mathrm{Gen}(A)$.

Next we state several propositions:

(25) Let $S$ be a non void non empty many sorted signature, and let $U_0$ be a non-empty algebra over $S$, and let $U_1$ be a subalgebra of $U_0$, and let $A$, $B$ be subsets of $U_0$. If $B = A \cup$ the sorts of $U_1$, then $\mathrm{Gen}(A) \bigsqcup U_1 = \mathrm{Gen}(B)$.

(26) Let $S$ be a non void non empty many sorted signature, and let $U_0$ be a non-empty algebra over $S$, and let $U_1$ be a subalgebra of $U_0$, and let $B$ be a subset of $U_0$. If $B =$ the sorts of $U_0$, then $\mathrm{Gen}(B) \bigsqcup U_1 = \mathrm{Gen}(B)$.

(27) Let $S$ be a non void non empty many sorted signature, and let $U_0$ be a non-empty algebra over $S$, and let $U_1$, $U_2$ be subalgebras of $U_0$. Then $U_1 \bigsqcup U_2 = U_2 \bigsqcup U_1$.

(28) Let $S$ be a non void non empty many sorted signature, and let $U_0$ be a non-empty algebra over $S$, and let $U_1$, $U_2$ be strict subalgebras of $U_0$. Then $U_1 \cap (U_1 \bigsqcup U_2) = U_1$.

(29) Let $S$ be a non void non empty many sorted signature with constant operations, and let $U_0$ be a non-empty algebra over $S$, and let $U_1$, $U_2$ be strict subalgebras of $U_0$. Then $U_1 \cap U_2 \bigsqcup U_2 = U_2$.

## 6. Lattice of Subalgebras of Many Sorted Algebra

Let $S$ be a non void non empty many sorted signature and let $U_0$ be an algebra over $S$. The functor Subalgebras($U_0$) yielding a non empty set is defined as follows:

(Def.20) For every $x$ holds $x \in$ Subalgebras($U_0$) iff $x$ is a strict subalgebra of $U_0$.

Let $S$ be a non void non empty many sorted signature and let $U_0$ be a non-empty algebra over $S$. The functor MSAlgJoin($U_0$) yields a binary operation on Subalgebras($U_0$) and is defined by:

(Def.21) For all elements $x$, $y$ of Subalgebras($U_0$) and for all strict subalgebras $U_1$, $U_2$ of $U_0$ such that $x = U_1$ and $y = U_2$ holds (MSAlgJoin($U_0$))($x$, $y$) $= U_1 \bigsqcup U_2$.

Let $S$ be a non void non empty many sorted signature and let $U_0$ be a non-empty algebra over $S$. The functor MSAlgMeet($U_0$) yielding a binary operation on Subalgebras($U_0$) is defined by:

(Def.22) For all elements $x$, $y$ of Subalgebras($U_0$) and for all strict subalgebras $U_1$, $U_2$ of $U_0$ such that $x = U_1$ and $y = U_2$ holds (MSAlgMeet($U_0$))($x$, $y$) $= U_1 \cap U_2$.

In the sequel $U_0$ is a non-empty algebra over $S$.

We now state four propositions:

(30) MSAlgJoin($U_0$) is commutative.

(31) MSAlgJoin($U_0$) is associative.

(32) Let $S$ be a non void non empty many sorted signature with constant operations and let $U_0$ be a non-empty algebra over $S$. Then MSAlgMeet($U_0$) is commutative.

(33) Let $S$ be a non void non empty many sorted signature with constant operations and let $U_0$ be a non-empty algebra over $S$. Then MSAlgMeet($U_0$) is associative.

Let $S$ be a non void non empty many sorted signature with constant operations and let $U_0$ be a non-empty algebra over $S$. The lattice of subalgebras of $U_0$ yields a strict lattice and is defined as follows:

(Def.23) The lattice of subalgebras of $U_0 = \langle$Subalgebras($U_0$), MSAlgJoin($U_0$), MSAlgMeet($U_0$)$\rangle$.

The following proposition is true

(34) Let $S$ be a non void non empty many sorted signature with constant operations and let $U_0$ be a non-empty algebra over $S$. Then the lattice of subalgebras of $U_0$ is bounded.

Let $S$ be a non void non empty many sorted signature with constant operations and let $U_0$ be a non-empty algebra over $S$. Note that the lattice of subalgebras of $U_0$ is bounded.

We now state three propositions:

(35) Let $S$ be a non void non empty many sorted signature with constant operations and let $U_0$ be a non-empty algebra over $S$. Then $\bot_{\text{the lattice of subalgebras of } U_0} = \mathrm{Gen}(\mathrm{Constants}(U_0))$.

(36) Let $S$ be a non void non empty many sorted signature with constant operations, and let $U_0$ be a non-empty algebra over $S$, and let $B$ be a subset of $U_0$. If $B =$ the sorts of $U_0$, then $\top_{\text{the lattice of subalgebras of } U_0} = \mathrm{Gen}(B)$.

(37) Let $S$ be a non void non empty many sorted signature with constant operations and let $U_0$ be a strict non-empty algebra over $S$. Then $\top_{\text{the lattice of subalgebras of } U_0} = U_0$.

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