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Products of Many Sorted Algebras

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Summary. Product of two many sorted universal algebras and product of family of many sorted universal algebras are defined in this article. Operations on functions, such that commute, Frege, are also introduced.

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The papers [17], [18], [9], [10], [6], [7], [13], [11], [14], [4], [8], [2], [1], [3], [5], [16], [12], and [15] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity we follow the rules: I, J denote sets, A, B denote many sorted sets of I, i, j, x are arbitrary, and S denotes a non empty many sorted signature.

A set has common domain if:

(Def.1) For all functions f, g such that $f \in$ it and $g \in$ it holds $\operatorname{dom} f = \operatorname{dom} g$.

Let us mention that there exists a set which is functional and non empty and has common domain.

The following proposition is true

(1) $\{\emptyset\}$ is a functional set with common domain.

Let X be a functional set with common domain. The functor $\mathrm{DOM}(X)$ yielding a set is defined as follows:

(Def.2) (i) For every function x such that $x \in X$ holds $\mathrm{DOM}(X) = \operatorname{dom} x$ if $X \neq \emptyset$,

(ii) $\mathrm{DOM}(X) = \emptyset$, otherwise.

We now state the proposition

(2) For every functional set X with common domain such that $X = \{\emptyset\}$ holds $\mathrm{DOM}(X) = \emptyset$.

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Let I be a set and let M be a non-empty many sorted set of I. Observe that $\prod M$ is functional and non empty and has common domain.

2. Operations on Functions

The scheme *LambdaDMS* deals with a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

There exists a many sorted set X of $\mathcal{A}$ such that for every element d of $\mathcal{A}$ holds $X(d) = \mathcal{F}(d)$

for all values of the parameters.

Let f be a function. The functor commute(f) yields a function yielding function and is defined as follows:

(Def.5)[1] $\mathrm{commute}(f) = \mathrm{curry}'\,\mathrm{uncurry}\,f$.

We now state several propositions:

(3) For every function f and for arbitrary x such that $x \in \mathrm{dom}\,\mathrm{commute}(f)$ holds $(\mathrm{commute}(f))(x)$ is a function.

(4) For all sets A, B, C and for every function f such that $A \neq \emptyset$ and $B \neq \emptyset$ and $f \in (C^B)^A$ holds $\mathrm{commute}(f) \in (C^A)^B$.

(5) Let A, B, C be sets and let f be a function. Suppose $A \neq \emptyset$ and $B \neq \emptyset$ and $f \in (C^B)^A$. Let g, h be functions and let x, y be arbitrary. Suppose $x \in A$ and $y \in B$ and $f(x) = g$ and $(\mathrm{commute}(f))(y) = h$. Then $h(x) = g(y)$ and $\mathrm{dom}\,h = A$ and $\mathrm{dom}\,g = B$ and $\mathrm{rng}\,h \subseteq C$ and $\mathrm{rng}\,g \subseteq C$.

(6) For all sets A, B, C and for every function f such that $A \neq \emptyset$ and $B \neq \emptyset$ and $f \in (C^B)^A$ holds $\mathrm{commute}(\mathrm{commute}(f)) = f$.

(7) $\mathrm{commute}(\square) = \square$.

Let F be a function. The functor $\blacksquare\mathrm{commute}(F)$ yielding a function is defined by the conditions (Def.6).

(Def.6) (i) For every x holds $x \in \mathrm{dom}\,\blacksquare\mathrm{commute}(F)$ iff there exists a function f such that $f \in \mathrm{dom}\,F$ and $x = \mathrm{commute}(f)$, and

(ii) for every function f such that $f \in \mathrm{dom}\,\blacksquare\mathrm{commute}(F)$ holds $(\blacksquare\mathrm{commute}(F))(f) = F(\mathrm{commute}(f))$.

The following proposition is true

(8) For every function F such that $\mathrm{dom}\,F = \{\emptyset\}$ holds $\blacksquare\mathrm{commute}(F) = F$.

Let F be a function yielding function and let f be a function. The functor $F \looparrowleft f$ yielding a function is defined by:

(Def.7) $\mathrm{dom}(F \looparrowleft f) = \mathrm{dom}\,F$ and for arbitrary x and for every function g such that $x \in \mathrm{dom}\,F$ and $g = F(x)$ holds $(F \looparrowleft f)(x) = g(f(x))$.

Let f be a function yielding function. The functor Frege(f) yields a many sorted function of $\prod(\mathrm{dom}_\kappa\, f(\kappa))$ and is defined as follows:

[1] The definitions (Def.3) and (Def.4) have been removed.

(Def.8) For every function g such that $g \in \prod(\mathrm{dom}_\kappa f(\kappa))$ holds $(\mathrm{Frege}(f))(g) = f \looparrowleft g$.

Let us consider I, A, B. The functor $[\![A, B]\!]$ yielding a many sorted set of I is defined by:

(Def.9) For every i such that $i \in I$ holds $[\![A, B]\!](i) = [\!: A(i),\ B(i) :\!]$.

Let us consider I and let A, B be non-empty many sorted sets of I. Note that $[\![A, B]\!]$ is non-empty.

Next we state the proposition

(9) Let I be a non empty set, and let J be a set, and let A, B be many sorted sets of I, and let f be a function from J into I. Then $[\![A, B]\!] \cdot f = [\![A \cdot f, B \cdot f]\!]$.

Let I be a non empty set, let us consider J, let A, B be non-empty many sorted sets of I, let p be a function from J into I^*, let r be a function from J into I, let j be arbitrary, let f be a function from $(A^\# \cdot p)(j)$ into $(A \cdot r)(j)$, and let g be a function from $(B^\# \cdot p)(j)$ into $(B \cdot r)(j)$. Let us assume that $j \in J$. The functor $]\!] f, g [\![$ yields a function from $([\![A, B]\!]^\# \cdot p)(j)$ into $([\![A, B]\!] \cdot r)(j)$ and is defined as follows:

(Def.10) For every function h such that $h \in ([\![A, B]\!]^\# \cdot p)(j)$ holds $]\!] f, g [\![(h) = \langle f(\mathrm{pr1}(h)),\ g(\mathrm{pr2}(h)) \rangle$.

Let I be a non empty set, let us consider J, let A, B be non-empty many sorted sets of I, let p be a function from J into I^*, let r be a function from J into I, let F be a many sorted function from $A^\# \cdot p$ into $A \cdot r$, and let G be a many sorted function from $B^\# \cdot p$ into $B \cdot r$. The functor $]\!] F, G [\![$ yielding a many sorted function from $[\![A, B]\!]^\# \cdot p$ into $[\![A, B]\!] \cdot r$ is defined by the condition (Def.11).

(Def.11) Given j. Suppose $j \in J$. Let f be a function from $(A^\# \cdot p)(j)$ into $(A \cdot r)(j)$ and let g be a function from $(B^\# \cdot p)(j)$ into $(B \cdot r)(j)$. If $f = F(j)$ and $g = G(j)$, then $]\!] F, G [\![(j) =]\!] f, g [\![$.

3. Family of Many Sorted Universal Algebras

Let us consider I, S. A many sorted set of I is said to be an algebra family of I over S if:

(Def.12) For every i such that $i \in I$ holds $\mathrm{it}(i)$ is a non-empty algebra over S.

Let I be a non empty set, let us consider S, let A be an algebra family of I over S, and let i be an element of I. Then $A(i)$ is a non-empty algebra over S.

Let S be a non empty many sorted signature and let U_1 be a non-empty algebra over S. The functor $|U_1|$ yields a non empty set and is defined as follows:

(Def.13) $|U_1| = \bigcup \mathrm{rng}\,(\text{the sorts of } U_1)$.

Let I be a non empty set, let S be a non empty many sorted signature, and let A be an algebra family of I over S. The functor $|A|$ yields a non empty set and is defined as follows:

(Def.14) $|A| = \bigcup\{|A(i)| : i$ ranges over elements of $I\}$.

4. Product of Many Sorted Universal Algebras

We now state two propositions:

(10) Let S be a non void non empty many sorted signature, and let U_0 be an algebra over S, and let o be an operation symbol of S. Then $\mathrm{Args}(o, U_0) = \prod((\text{the sorts of } U_0) \cdot \mathrm{Arity}(o))$ and $\mathrm{dom}((\text{the sorts of } U_0) \cdot \mathrm{Arity}(o)) = \mathrm{dom}\,\mathrm{Arity}(o)$ and $\mathrm{Result}(o, U_0) = (\text{the sorts of } U_0)(\text{the result sort of } o)$.

(11) Let S be a non void non empty many sorted signature, and let U_0 be an algebra over S, and let o be an operation symbol of S. If $\mathrm{Arity}(o) = \varepsilon$, then $\mathrm{Args}(o, U_0) = \{\square\}$.

Let us consider S and let U_1, U_2 be non-empty algebras over S. The functor $[\!: U_1, U_2 :\!]$ yields a strict algebra over S and is defined as follows:

(Def.15) $[\!: U_1, U_2 :\!] = \langle [\![\text{the sorts of } U_1, \text{the sorts of } U_2]\!],]\!]\text{the characteristics of } U_1, (\text{the characteristics of } U_2)[\![\rangle$.

Let I be a non empty set, let us consider S, let s be a sort symbol of S, and let A be an algebra family of I over S. The functor $\mathrm{Carrier}(A, s)$ yielding a non-empty many sorted set of I is defined as follows:

(Def.16) For every element i of I holds $(\mathrm{Carrier}(A, s))(i) = (\text{the sorts of } A(i))(s)$.

Let I be a non empty set, let us consider S, and let A be an algebra family of I over S. The functor $\mathrm{SORTS}(A)$ yields a non-empty many sorted set of the carrier of S and is defined as follows:

(Def.17) For every sort symbol s of S holds $(\mathrm{SORTS}(A))(s) = \prod \mathrm{Carrier}(A, s)$.

Let I be a non empty set, let S be a non empty many sorted signature, and let A be an algebra family of I over S. The functor $\mathrm{OPER}(A)$ yields a many sorted function of I and is defined by:

(Def.18) For every element i of I holds $(\mathrm{OPER}(A))(i) =$ the characteristics of $A(i)$.

We now state two propositions:

(12) Let I be a non empty set, and let S be a non empty many sorted signature, and let A be an algebra family of I over S. Then $\mathrm{dom}\,\mathrm{uncurry}\,\mathrm{OPER}(A) = [\!: I, \text{the operation symbols of } S :\!]$.

(13) Let I be a non empty set, and let S be a non void non empty many sorted signature, and let A be an algebra family of I over S, and let o be an operation symbol of S. Then $\mathrm{commute}(\mathrm{OPER}(A)) \in ((\mathrm{rng}\,\mathrm{uncurry}\,\mathrm{OPER}(A))^I)^{\text{the operation symbols of } S}$.

Let I be a non empty set, let S be a non void non empty many sorted signature, let A be an algebra family of I over S, and let o be an operation symbol of S. The functor $A(o)$ yielding a many sorted function of I is defined by:

(Def.19) $A(o) = (\mathrm{commute}(\mathrm{OPER}(A)))(o)$.

We now state several propositions:

(14) Let I be a non empty set, and let i be an element of I, and let S be a non void non empty many sorted signature, and let A be an algebra family of I over S, and let o be an operation symbol of S. Then $A(o)(i) = \mathrm{Den}(o, A(i))$.

(15) Let I be a non empty set, and let S be a non void non empty many sorted signature, and let A be an algebra family of I over S, and let o be an operation symbol of S, and let x be arbitrary. If $x \in \mathrm{rng}\,\mathrm{Frege}(A(o))$, then x is a function.

(16) Let I be a non empty set, and let S be a non void non empty many sorted signature, and let A be an algebra family of I over S, and let o be an operation symbol of S, and let f be a function. If $f \in \mathrm{rng}\,\mathrm{Frege}(A(o))$, then $\mathrm{dom}\, f = I$ and for every element i of I holds $f(i) \in \mathrm{Result}(o, A(i))$.

(17) Let I be a non empty set, and let S be a non void non empty many sorted signature, and let A be an algebra family of I over S, and let o be an operation symbol of S, and let f be a function. Suppose $f \in \mathrm{dom}\,\mathrm{Frege}(A(o))$. Then $\mathrm{dom}\, f = I$ and for every element i of I holds $f(i) \in \mathrm{Args}(o, A(i))$ and $\mathrm{rng}\, f \subseteq |A|^{\mathrm{dom}\,\mathrm{Arity}(o)}$.

(18) Let I be a non empty set, and let S be a non void non empty many sorted signature, and let A be an algebra family of I over S, and let o be an operation symbol of S. Then $\mathrm{dom}(\mathrm{dom}_\kappa\, A(o)(\kappa)) = I$ and for every element i of I holds $(\mathrm{dom}_\kappa\, A(o)(\kappa))(i) = \mathrm{Args}(o, A(i))$.

Let I be a non empty set, let S be a non void non empty many sorted signature, and let A be an algebra family of I over S. The functor $\mathrm{OPS}(A)$ yielding a many sorted function from $(\mathrm{SORTS}(A))^{\#} \cdot (\text{the arity of } S)$ into $\mathrm{SORTS}(A) \cdot (\text{the result sort of } S)$ is defined by:

(Def.20) For every operation symbol o of S holds $(\mathrm{OPS}(A))(o) = (\mathrm{Arity}(o) = \varepsilon \rightarrow \mathrm{commute}(A(o)), \blacksquare \mathrm{commute}(\mathrm{Frege}(A(o))))$.

Let I be a non empty set, let S be a non void non empty many sorted signature, and let A be an algebra family of I over S. The functor $\prod A$ yields a strict algebra over S and is defined as follows:

(Def.21) $\prod A = \langle \mathrm{SORTS}(A), \mathrm{OPS}(A) \rangle$.

We now state two propositions:

(19) Let I be a non empty set, and let S be a non void non empty many sorted signature, and let A be an algebra family of I over S. Then $\prod A = \langle \mathrm{SORTS}(A), \mathrm{OPS}(A) \rangle$.

(20) Let I be a non empty set, and let S be a non void non empty many sorted signature, and let A be an algebra family of I over S. Then the

sorts of $\prod A = \mathrm{SORTS}(A)$ and the characteristics of $\prod A = \mathrm{OPS}(A)$.

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