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Continuous, Stable, and Linear Maps of Coherence Spaces

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The papers [18], [21], [9], [14], [16], [11], [3], [19], [22], [7], [6], [10], [20], [12], [13], [17], [1], [2], [5], [8], [15], and [4] provide the terminology and notation for this paper.

1. Directed Sets

One can check that there exists a coherent space which is finite. Let us observe that a set is binary complete if:

(Def.1) For every set A such that for all sets a, b such that $a \in A$ and $b \in A$ holds $a \cup b \in$ it holds $\bigcup A \in$ it.

Let X be a set. The functor $\mathrm{FlatCoh}(X)$ yielding a set is defined as follows:

(Def.2) $\mathrm{FlatCoh}(X) = \mathrm{CohSp}(\triangle_X)$.

The functor $\mathrm{SubFin}(X)$ yielding a subset of X is defined by:

(Def.3) For every set x holds $x \in \mathrm{SubFin}(X)$ iff $x \in X$ and x is finite.

One can prove the following three propositions:

(1) For all sets X, x holds $x \in \mathrm{FlatCoh}(X)$ iff $x = \emptyset$ or there exists a set y such that $x = \{y\}$ and $y \in X$.

(2) For every set X holds $\bigcup \mathrm{FlatCoh}(X) = X$.

(3) For every finite down-closed set X holds $\mathrm{SubFin}(X) = X$.

One can check that $\{\emptyset\}$ is down-closed and binary complete. Let X be a set. One can check that 2^X is down-closed and binary complete and $\mathrm{FlatCoh}(X)$ is non empty down-closed and binary complete.

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Let C be a non empty down-closed set. Observe that SubFin(C) is non empty and down-closed.

We now state the proposition

(4) Web($\{\emptyset\}$) $= \emptyset$.

The scheme *MinimalElement wrt Incl* concerns sets $\mathcal{A}$, $\mathcal{B}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a set a such that $a \in \mathcal{B}$ and $\mathcal{P}[a]$ and for every set b such that $b \in \mathcal{B}$ and $\mathcal{P}[b]$ and $b \subseteq a$ holds $b = a$

provided the following requirements are met:

- $\mathcal{A} \in \mathcal{B}$,
- $\mathcal{P}[\mathcal{A}]$,
- $\mathcal{A}$ is finite.

Let X be a set. One can check that there exists a subset of X which is finite.

Let C be a coherent space. Observe that there exists an element of C which is finite.

Let X be a set. We say that X is $\cup$-directed if and only if:

(Def.4) For every finite subset Y of X there exists a set a such that $\bigcup Y \subseteq a$ and $a \in X$.

We say that X is $\cap$-directed if and only if:

(Def.5) For every finite subset Y of X there exists a set a such that for every set y such that $y \in Y$ holds $a \subseteq y$ and $a \in X$.

Let us note that every set which is $\cup$-directed is also non empty and every set which is $\cap$-directed is also non empty.

We now state several propositions:

(5) Let X be a set. Suppose X is $\cup$-directed. Let a, b be sets. If $a \in X$ and $b \in X$, then there exists a set c such that $a \cup b \subseteq c$ and $c \in X$.

(6) Let X be a non empty set. Suppose that for all sets a, b such that $a \in X$ and $b \in X$ there exists a set c such that $a \cup b \subseteq c$ and $c \in X$. Then X is $\cup$-directed.

(7) Let X be a set. Suppose X is $\cap$-directed. Let a, b be sets. If $a \in X$ and $b \in X$, then there exists a set c such that $c \subseteq a \cap b$ and $c \in X$.

(8) Let X be a non empty set. Suppose that for all sets a, b such that $a \in X$ and $b \in X$ there exists a set c such that $c \subseteq a \cap b$ and $c \in X$. Then X is $\cap$-directed.

(9) For every set x holds $\{x\}$ is $\cup$-directed and $\cap$-directed.

(10) For all sets x, y holds $\{x, y, x \cup y\}$ is $\cup$-directed.

(11) For all sets x, y holds $\{x, y, x \cap y\}$ is $\cap$-directed.

Let us observe that there exists a set which is $\cup$-directed $\cap$-directed and finite.

Let C be a non empty set. Observe that there exists a subset of C which is $\cup$-directed $\cap$-directed and finite.

We now state the proposition

(12) For every set X holds $\operatorname{Fin} X$ is $\cup$-directed and $\cap$-directed.

Let X be a set. Observe that $\operatorname{Fin} X$ is $\cup$-directed and $\cap$-directed.

Let C be a down-closed non empty set. Note that there exists a subset of C which is preboolean and non empty.

Let C be a down-closed non empty set and let a be an element of C. Then $\operatorname{Fin} a$ is a preboolean non empty subset of C.

One can prove the following proposition

(13) Let X be a non empty set and let Y be a set. Suppose X is $\cup$-directed and $Y \subseteq \bigcup X$ and Y is finite. Then there exists a set Z such that $Z \in X$ and $Y \subseteq Z$.

Let X be a set. We say that X is $\cap$-closed if and only if:

(Def.6) For all sets x, y such that $x \in X$ and $y \in X$ holds $x \cap y \in X$.

We say that X is closed under directed unions if and only if:

(Def.7) For every subset A of X such that A is $\cup$-directed holds $\bigcup A \in X$.

One can check that every set which is down-closed is also $\cap$-closed.

Next we state two propositions:

(14) For every coherent space C and for all elements x, y of C holds $x \cap y \in C$.

(15) For every coherent space C and for every $\cup$-directed subset A of C holds $\bigcup A \in C$.

Let us note that every coherent space is closed under directed unions.

Let us note that there exists a coherent space which is $\cap$-closed and closed under directed unions.

Let C be a closed under directed unions non empty set and let A be a $\cup$-directed subset of C. Then $\bigcup A$ is an element of C.

Let X, Y be sets. We say that X includes lattice of Y if and only if:

(Def.8) For all sets a, b such that $a \in Y$ and $b \in Y$ holds $a \cap b \in X$ and $a \cup b \in X$.

The following proposition is true

(16) For every non empty set X such that X includes lattice of X holds X is $\cup$-directed and $\cap$-directed.

Let X, x, y be sets. We say that X includes lattice of x, y if and only if:

(Def.9) X includes lattice of $\{x, y\}$.

One can prove the following proposition

(17) For all sets X, x, y holds X includes lattice of x, y iff $x \in X$ and $y \in X$ and $x \cap y \in X$ and $x \cup y \in X$.

2. Continuous, Stable, and Linear Functions

Let f be a function. We say that f is preserving arbitrary unions if and only if:

(Def.10) For every subset A of $\operatorname{dom} f$ such that $\bigcup A \in \operatorname{dom} f$ holds $f(\bigcup A) = \bigcup(f^\circ A)$.

We say that f is preserving directed unions if and only if:

(Def.11) For every subset A of $\mathrm{dom}\, f$ such that A is $\cup$-directed and $\bigcup A \in \mathrm{dom}\, f$ holds $f(\bigcup A) = \bigcup(f^{\circ}A)$.

Let f be a function. We say that f is $\subseteq$-monotone if and only if:

(Def.12) For all sets a, b such that $a \in \mathrm{dom}\, f$ and $b \in \mathrm{dom}\, f$ and $a \subseteq b$ holds $f(a) \subseteq f(b)$.

We say that f is preserving binary intersections if and only if:

(Def.13) For all sets a, b such that $\mathrm{dom}\, f$ includes lattice of a, b holds $f(a \cap b) = f(a) \cap f(b)$.

Let us note that every function which is preserving directed unions is also $\subseteq$-monotone and every function which is preserving arbitrary unions is also preserving directed unions.

Next we state two propositions:

(18) Let f be a function. Suppose f is preserving arbitrary unions. Let x, y be sets. If $x \in \mathrm{dom}\, f$ and $y \in \mathrm{dom}\, f$ and $x \cup y \in \mathrm{dom}\, f$, then $f(x \cup y) = f(x) \cup f(y)$.

(19) For every function f such that f is preserving arbitrary unions holds $f(\emptyset) = \emptyset$.

Let C_1, C_2 be coherent spaces. Note that there exists a function from C_1 into C_2 which is preserving arbitrary unions and preserving binary intersections.

Let C be a coherent space. One can verify that there exists a many sorted set indexed by C which is preserving arbitrary unions and preserving binary intersections.

Let f be a function. We say that f is continuous if and only if:

(Def.14) $\mathrm{dom}\, f$ is closed under directed unions and f is preserving directed unions.

Let f be a function. We say that f is stable if and only if:

(Def.15) $\mathrm{dom}\, f$ is $\cap$-closed and f is continuous and preserving binary intersections.

Let f be a function. We say that f is linear if and only if:

(Def.16) f is stable and preserving arbitrary unions.

One can check the following observations:

* every function which is continuous is also preserving directed unions,
* every function which is stable is also preserving binary intersections and continuous, and
* every function which is linear is also preserving arbitrary unions and stable.

Let X be a closed under directed unions set. Note that every many sorted set indexed by X which is preserving directed unions is also continuous.

Let X be a $\cap$-closed set. Observe that every many sorted set indexed by X which is continuous and preserving binary intersections is also stable.

Let us note that every function which is stable and preserving arbitrary unions is also linear.

Note that there exists a function which is linear. Let C be a coherent space. One can check that there exists a many sorted set indexed by C which is linear. Let B be a coherent space. One can check that there exists a function from B into C which is linear.

Let f be a continuous function. One can verify that $\operatorname{dom} f$ is closed under directed unions.

Let f be a stable function. One can verify that $\operatorname{dom} f$ is $\cap$-closed.

We now state several propositions:

(20) For every set X holds $\bigcup \operatorname{Fin} X = X$.

(21) For every continuous function f such that $\operatorname{dom} f$ is down-closed and for every set a such that $a \in \operatorname{dom} f$ holds $f(a) = \bigcup (f^\circ \operatorname{Fin} a)$.

(22) Let f be a function. Suppose $\operatorname{dom} f$ is down-closed. Then f is continuous if and only if the following conditions are satisfied:

(i) $\operatorname{dom} f$ is closed under directed unions,

(ii) f is $\subseteq$-monotone, and

(iii) for all sets a, y such that $a \in \operatorname{dom} f$ and $y \in f(a)$ there exists a set b such that b is finite and $b \subseteq a$ and $y \in f(b)$.

(23) Let f be a function. Suppose $\operatorname{dom} f$ is down-closed and closed under directed unions. Then f is stable if and only if the following conditions are satisfied:

(i) f is $\subseteq$-monotone, and

(ii) for all sets a, y such that $a \in \operatorname{dom} f$ and $y \in f(a)$ there exists a set b such that b is finite and $b \subseteq a$ and $y \in f(b)$ and for every set c such that $c \subseteq a$ and $y \in f(c)$ holds $b \subseteq c$.

(24) Let f be a function. Suppose $\operatorname{dom} f$ is down-closed and closed under directed unions. Then f is linear if and only if the following conditions are satisfied:

(i) f is $\subseteq$-monotone, and

(ii) for all sets a, y such that $a \in \operatorname{dom} f$ and $y \in f(a)$ there exists a set x such that $x \in a$ and $y \in f(\{x\})$ and for every set b such that $b \subseteq a$ and $y \in f(b)$ holds $x \in b$.

3. Graph of Continuous Function

Let f be a function. The functor $\operatorname{graph}(f)$ yielding a set is defined as follows:

(Def.17) For every set x holds $x \in \operatorname{graph}(f)$ iff there exists a finite set y and there exists a set z such that $x = \langle y, z \rangle$ and $y \in \operatorname{dom} f$ and $z \in f(y)$.

Let C_1, C_2 be non empty sets and let f be a function from C_1 into C_2. Then $\operatorname{graph}(f)$ is a subset of $[\!: C_1, \bigcup C_2 :\!]$.

Let f be a function. Note that $\operatorname{graph}(f)$ is relation-like.

Next we state several propositions:

(25) For every function f and for all sets x, y holds $\langle x, y\rangle \in \mathrm{graph}(f)$ iff x is finite and $x \in \mathrm{dom}\, f$ and $y \in f(x)$.

(26) Let f be a $\subseteq$-monotone function and let a, b be sets. Suppose $b \in \mathrm{dom}\, f$ and $a \subseteq b$ and b is finite. Let y be a set. If $\langle a, y\rangle \in \mathrm{graph}(f)$, then $\langle b, y\rangle \in \mathrm{graph}(f)$.

(27) Let C_1, C_2 be coherent spaces, and let f be a function from C_1 into C_2, and let a be an element of C_1, and let y_1, y_2 be sets. If $\langle a, y_1\rangle \in \mathrm{graph}(f)$ and $\langle a, y_2\rangle \in \mathrm{graph}(f)$, then $\{y_1, y_2\} \in C_2$.

(28) Let C_1, C_2 be coherent spaces, and let f be a $\subseteq$-monotone function from C_1 into C_2, and let a, b be elements of C_1. Suppose $a \cup b \in C_1$. Let y_1, y_2 be sets. If $\langle a, y_1\rangle \in \mathrm{graph}(f)$ and $\langle b, y_2\rangle \in \mathrm{graph}(f)$, then $\{y_1, y_2\} \in C_2$.

(29) For all coherent spaces C_1, C_2 and for all continuous functions f, g from C_1 into C_2 such that $\mathrm{graph}(f) = \mathrm{graph}(g)$ holds $f = g$.

(30) Let C_1, C_2 be coherent spaces and let X be a subset of $[\!:C_1, \bigcup C_2:\!]$. Suppose that

(i) for every set x such that $x \in X$ holds $x_{\mathbf{1}}$ is finite,

(ii) for all finite elements a, b of C_1 such that $a \subseteq b$ and for every set y such that $\langle a, y\rangle \in X$ holds $\langle b, y\rangle \in X$, and

(iii) for every finite element a of C_1 and for all sets y_1, y_2 such that $\langle a, y_1\rangle \in X$ and $\langle a, y_2\rangle \in X$ holds $\{y_1, y_2\} \in C_2$.

Then there exists a continuous function f from C_1 into C_2 such that $X = \mathrm{graph}(f)$.

(31) Let C_1, C_2 be coherent spaces, and let f be a continuous function from C_1 into C_2, and let a be an element of C_1. Then $f(a) = (\mathrm{graph}(f))^{\circ}\,\mathrm{Fin}\, a$.

4. Trace of Stable Function

Let f be a function. The functor $\mathrm{Trace}(f)$ yields a set and is defined by the condition (Def.18).

(Def.18) Let x be a set. Then $x \in \mathrm{Trace}(f)$ if and only if there exist sets a, y such that $x = \langle a, y\rangle$ and $a \in \mathrm{dom}\, f$ and $y \in f(a)$ and for every set b such that $b \in \mathrm{dom}\, f$ and $b \subseteq a$ and $y \in f(b)$ holds $a = b$.

Next we state the proposition

(32) Let f be a function and let a, y be sets. Then $\langle a, y\rangle \in \mathrm{Trace}(f)$ if and only if the following conditions are satisfied:

(i) $a \in \mathrm{dom}\, f$,

(ii) $y \in f(a)$, and

(iii) for every set b such that $b \in \mathrm{dom}\, f$ and $b \subseteq a$ and $y \in f(b)$ holds $a = b$.

Let C_1, C_2 be non empty sets and let f be a function from C_1 into C_2. Then $\mathrm{Trace}(f)$ is a subset of $[\!:C_1, \bigcup C_2:\!]$.

Let f be a function. One can check that $\mathrm{Trace}(f)$ is relation-like.

Next we state a number of propositions:

(33) For every continuous function f such that $\mathrm{dom}\, f$ is down-closed holds $\mathrm{Trace}(f) \subseteq \mathrm{graph}(f)$.

(34) Let f be a continuous function. Suppose $\mathrm{dom}\, f$ is down-closed. Let a, y be sets. If $\langle a, y \rangle \in \mathrm{Trace}(f)$, then a is finite.

(35) Let C_1, C_2 be coherent spaces, and let f be a $\subseteq$-monotone function from C_1 into C_2, and let a_1, a_2 be sets. Suppose $a_1 \cup a_2 \in C_1$. Let y_1, y_2 be sets. If $\langle a_1, y_1 \rangle \in \mathrm{Trace}(f)$ and $\langle a_2, y_2 \rangle \in \mathrm{Trace}(f)$, then $\{y_1, y_2\} \in C_2$.

(36) Let C_1, C_2 be coherent spaces, and let f be a preserving binary intersections function from C_1 into C_2, and let a_1, a_2 be sets. If $a_1 \cup a_2 \in C_1$, then for every set y such that $\langle a_1, y \rangle \in \mathrm{Trace}(f)$ and $\langle a_2, y \rangle \in \mathrm{Trace}(f)$ holds $a_1 = a_2$.

(37) Let C_1, C_2 be coherent spaces and let f, g be stable functions from C_1 into C_2. If $\mathrm{Trace}(f) \subseteq \mathrm{Trace}(g)$, then for every element a of C_1 holds $f(a) \subseteq g(a)$.

(38) For all coherent spaces C_1, C_2 and for all stable functions f, g from C_1 into C_2 such that $\mathrm{Trace}(f) = \mathrm{Trace}(g)$ holds $f = g$.

(39) Let C_1, C_2 be coherent spaces and let X be a subset of $[\!: C_1, \bigcup C_2 :\!]$. Suppose that

(i) for every set x such that $x \in X$ holds $x_{\mathbf{1}}$ is finite,

(ii) for all elements a, b of C_1 such that $a \cup b \in C_1$ and for all sets y_1, y_2 such that $\langle a, y_1 \rangle \in X$ and $\langle b, y_2 \rangle \in X$ holds $\{y_1, y_2\} \in C_2$, and

(iii) for all elements a, b of C_1 such that $a \cup b \in C_1$ and for every set y such that $\langle a, y \rangle \in X$ and $\langle b, y \rangle \in X$ holds $a = b$.

Then there exists a stable function f from C_1 into C_2 such that $X = \mathrm{Trace}(f)$.

(40) Let C_1, C_2 be coherent spaces, and let f be a stable function from C_1 into C_2, and let a be an element of C_1. Then $f(a) = (\mathrm{Trace}(f))^{\circ}\, \mathrm{Fin}\, a$.

(41) Let C_1, C_2 be coherent spaces, and let f be a stable function from C_1 into C_2, and let a be an element of C_1, and let y be a set. Then $y \in f(a)$ if and only if there exists an element b of C_1 such that $\langle b, y \rangle \in \mathrm{Trace}(f)$ and $b \subseteq a$.

(42) For all coherent spaces C_1, C_2 there exists a stable function f from C_1 into C_2 such that $\mathrm{Trace}(f) = \emptyset$.

(43) Let C_1, C_2 be coherent spaces, and let a be a finite element of C_1, and let y be a set. If $y \in \bigcup C_2$, then there exists a stable function f from C_1 into C_2 such that $\mathrm{Trace}(f) = \{\langle a, y \rangle\}$.

(44) Let C_1, C_2 be coherent spaces, and let a be an element of C_1, and let y be a set. Suppose $y \in \bigcup C_2$. Let f be a stable function from C_1 into C_2. Suppose $\mathrm{Trace}(f) = \{\langle a, y \rangle\}$. Let b be an element of C_1. Then if $a \subseteq b$, then $f(b) = \{y\}$ and if $a \not\subseteq b$, then $f(b) = \emptyset$.

(45) Let C_1, C_2 be coherent spaces, and let f be a stable function from C_1 into C_2, and let X be a subset of Trace(f). Then there exists a stable function g from C_1 into C_2 such that Trace(g) $= X$.

(46) Let C_1, C_2 be coherent spaces and let A be a set. Suppose that for all sets x, y such that $x \in A$ and $y \in A$ there exists a stable function f from C_1 into C_2 such that $x \cup y =$ Trace(f). Then there exists a stable function f from C_1 into C_2 such that $\bigcup A =$ Trace(f).

Let C_1, C_2 be coherent spaces. The functor StabCoh(C_1, C_2) yielding a set is defined as follows:

(Def.19) For every set x holds $x \in$ StabCoh(C_1, C_2) iff there exists a stable function f from C_1 into C_2 such that $x =$ Trace(f).

Let C_1, C_2 be coherent spaces. Note that StabCoh(C_1, C_2) is non empty down-closed and binary complete.

We now state three propositions:

(47) For all coherent spaces C_1, C_2 and for every stable function f from C_1 into C_2 holds Trace(f) $\subseteq [\!:$ SubFin(C_1), $\bigcup C_2 :\!]$.

(48) For all coherent spaces C_1, C_2 holds $\bigcup$StabCoh(C_1, C_2) $= [\!:$ SubFin(C_1), $\bigcup C_2 :\!]$.

(49) Let C_1, C_2 be coherent spaces, and let a, b be finite elements of C_1, and let y_1, y_2 be sets. Then $\langle\langle a, y_1\rangle, \langle b, y_2\rangle\rangle \in$ Web(StabCoh(C_1, C_2)) if and only if one of the following conditions is satisfied:
  (i) $a \cup b \notin C_1$ and $y_1 \in \bigcup C_2$ and $y_2 \in \bigcup C_2$, or
  (ii) $\langle y_1, y_2\rangle \in$ Web(C_2) and if $y_1 = y_2$, then $a = b$.

5. Trace of Linear Function

The following proposition is true

(50) Let C_1, C_2 be coherent spaces and let f be a stable function from C_1 into C_2. Then f is linear if and only if for all sets a, y such that $\langle a, y\rangle \in$ Trace(f) there exists a set x such that $a = \{x\}$.

Let f be a function. The functor LinTrace(f) yielding a set is defined as follows:

(Def.20) For every set x holds $x \in$ LinTrace(f) iff there exist sets y, z such that $x = \langle y, z\rangle$ and $\langle\{y\}, z\rangle \in$ Trace(f).

Next we state three propositions:

(51) For every function f and for all sets x, y holds $\langle x, y\rangle \in$ LinTrace(f) iff $\langle\{x\}, y\rangle \in$ Trace(f).

(52) For every function f such that $f(\emptyset) = \emptyset$ and for all sets x, y such that $\{x\} \in \operatorname{dom} f$ and $y \in f(\{x\})$ holds $\langle x, y\rangle \in$ LinTrace(f).

(53) For every function f and for all sets x, y such that $\langle x, y\rangle \in$ LinTrace(f) holds $\{x\} \in \operatorname{dom} f$ and $y \in f(\{x\})$.

Let C_1, C_2 be non empty sets and let f be a function from C_1 into C_2. Then $\mathrm{LinTrace}(f)$ is a subset of $[: \bigcup C_1, \bigcup C_2 :]$.

Let f be a function. One can verify that $\mathrm{LinTrace}(f)$ is relation-like.

Let C_1, C_2 be coherent spaces. The functor $\mathrm{LinCoh}(C_1, C_2)$ yielding a set is defined as follows:

(Def.21) For every set x holds $x \in \mathrm{LinCoh}(C_1, C_2)$ iff there exists a linear function f from C_1 into C_2 such that $x = \mathrm{LinTrace}(f)$.

Next we state a number of propositions:

(54) Let C_1, C_2 be coherent spaces, and let f be a $\subseteq$-monotone function from C_1 into C_2, and let x_1, x_2 be sets. Suppose $\{x_1, x_2\} \in C_1$. Let y_1, y_2 be sets. If $\langle x_1, y_1 \rangle \in \mathrm{LinTrace}(f)$ and $\langle x_2, y_2 \rangle \in \mathrm{LinTrace}(f)$, then $\{y_1, y_2\} \in C_2$.

(55) Let C_1, C_2 be coherent spaces, and let f be a preserving binary intersections function from C_1 into C_2, and let x_1, x_2 be sets. If $\{x_1, x_2\} \in C_1$, then for every set y such that $\langle x_1, y \rangle \in \mathrm{LinTrace}(f)$ and $\langle x_2, y \rangle \in \mathrm{LinTrace}(f)$ holds $x_1 = x_2$.

(56) For all coherent spaces C_1, C_2 and for all linear functions f, g from C_1 into C_2 such that $\mathrm{LinTrace}(f) = \mathrm{LinTrace}(g)$ holds $f = g$.

(57) Let C_1, C_2 be coherent spaces and let X be a subset of $[: \bigcup C_1, \bigcup C_2 :]$. Suppose that

(i) for all sets a, b such that $\{a, b\} \in C_1$ and for all sets y_1, y_2 such that $\langle a, y_1 \rangle \in X$ and $\langle b, y_2 \rangle \in X$ holds $\{y_1, y_2\} \in C_2$, and

(ii) for all sets a, b such that $\{a, b\} \in C_1$ and for every set y such that $\langle a, y \rangle \in X$ and $\langle b, y \rangle \in X$ holds $a = b$.

Then there exists a linear function f from C_1 into C_2 such that $X = \mathrm{LinTrace}(f)$.

(58) Let C_1, C_2 be coherent spaces, and let f be a linear function from C_1 into C_2, and let a be an element of C_1. Then $f(a) = (\mathrm{LinTrace}(f))^\circ a$.

(59) For all coherent spaces C_1, C_2 there exists a linear function f from C_1 into C_2 such that $\mathrm{LinTrace}(f) = \emptyset$.

(60) Let C_1, C_2 be coherent spaces, and let x be a set, and let y be a set. Suppose $x \in \bigcup C_1$ and $y \in \bigcup C_2$. Then there exists a linear function f from C_1 into C_2 such that $\mathrm{LinTrace}(f) = \{\langle x, y \rangle\}$.

(61) Let C_1, C_2 be coherent spaces, and let x be a set, and let y be a set. Suppose $x \in \bigcup C_1$ and $y \in \bigcup C_2$. Let f be a linear function from C_1 into C_2. Suppose $\mathrm{LinTrace}(f) = \{\langle x, y \rangle\}$. Let a be an element of C_1. Then if $x \in a$, then $f(a) = \{y\}$ and if $x \notin a$, then $f(a) = \emptyset$.

(62) Let C_1, C_2 be coherent spaces, and let f be a linear function from C_1 into C_2, and let X be a subset of $\mathrm{LinTrace}(f)$. Then there exists a linear function g from C_1 into C_2 such that $\mathrm{LinTrace}(g) = X$.

(63) Let C_1, C_2 be coherent spaces and let A be a set. Suppose that for all sets x, y such that $x \in A$ and $y \in A$ there exists a linear function f

from C_1 into C_2 such that $x \cup y = \mathrm{LinTrace}(f)$. Then there exists a linear function f from C_1 into C_2 such that $\bigcup A = \mathrm{LinTrace}(f)$.

Let C_1, C_2 be coherent spaces. One can check that $\mathrm{LinCoh}(C_1, C_2)$ is non empty down-closed and binary complete.

One can prove the following propositions:

(64) For all coherent spaces C_1, C_2 holds $\bigcup \mathrm{LinCoh}(C_1, C_2) = [\!:\bigcup C_1, \bigcup C_2:\!]$.

(65) Let C_1, C_2 be coherent spaces, and let x_1, x_2 be sets, and let y_1, y_2 be sets. Then $\langle\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\rangle \in \mathrm{Web}(\mathrm{LinCoh}(C_1, C_2))$ if and only if the following conditions are satisfied:

(i) $x_1 \in \bigcup C_1$,

(ii) $x_2 \in \bigcup C_1$, and

(iii) $\langle x_1, x_2\rangle \notin \mathrm{Web}(C_1)$ and $y_1 \in \bigcup C_2$ and $y_2 \in \bigcup C_2$ or $\langle y_1, y_2\rangle \in \mathrm{Web}(C_2)$ and if $y_1 = y_2$, then $x_1 = x_2$.

6. Negation of Coherence Spaces

Let C be a coherent space. The functor $\neg C$ yielding a set is defined by:

(Def.22) $\neg C = \{a : a$ ranges over subsets of $\bigcup C$, $\bigwedge_{b\,:\,\text{element of } C} \bigvee_{x\,:\,\text{set}}\ a \cap b \subseteq \{x\}\}$.

One can prove the following proposition

(66) Let C be a coherent space and let x be a set. Then $x \in \neg C$ if and only if the following conditions are satisfied:

(i) $x \subseteq \bigcup C$, and

(ii) for every element a of C there exists a set z such that $x \cap a \subseteq \{z\}$.

Let C be a coherent space. Observe that $\neg C$ is non empty down-closed and binary complete.

Next we state several propositions:

(67) For every coherent space C holds $\bigcup \neg C = \bigcup C$.

(68) For every coherent space C and for all sets x, y such that $x \neq y$ and $\{x, y\} \in C$ holds $\{x, y\} \notin \neg C$.

(69) For every coherent space C and for all sets x, y such that $\{x, y\} \subseteq \bigcup C$ and $\{x, y\} \notin C$ holds $\{x, y\} \in \neg C$.

(70) For every coherent space C and for all sets x, y holds $\langle x, y\rangle \in \mathrm{Web}(\neg C)$ iff $x \in \bigcup C$ but $y \in \bigcup C$ but $x = y$ or $\langle x, y\rangle \notin \mathrm{Web}(C)$.

(71) For every coherent space C holds $\neg\neg C = C$.

(72) $\neg\{\emptyset\} = \{\emptyset\}$.

(73) For every set X holds $\neg\,\mathrm{FlatCoh}(X) = 2^X$ and $\neg(2^X) = \mathrm{FlatCoh}(X)$.

7. Product and Coproduct on Coherence Spaces

Let x, y be sets. The functor $x \uplus y$ yielding a set is defined by:

(Def.23) $x \uplus y = \bigcup \mathrm{disjoint}\langle x, y\rangle$.

We now state a number of propositions:

(74) For all sets x, y holds $x \uplus y = [\!: x, \{1\} :\!] \cup [\!: y, \{2\} :\!]$.

(75) For every set x holds $x \uplus \emptyset = [\!: x, \{1\} :\!]$ and $\emptyset \uplus x = [\!: x, \{2\} :\!]$.

(76) For all sets x, y, z such that $z \in x \uplus y$ holds $z = \langle z_{\mathbf{1}}, z_{\mathbf{2}}\rangle$ but $z_{\mathbf{2}} = 1$ and $z_{\mathbf{1}} \in x$ or $z_{\mathbf{2}} = 2$ and $z_{\mathbf{1}} \in y$.

(77) For all sets x, y, z holds $\langle z, 1\rangle \in x \uplus y$ iff $z \in x$.

(78) For all sets x, y, z holds $\langle z, 2\rangle \in x \uplus y$ iff $z \in y$.

(79) For all sets x_1, y_1, x_2, y_2 holds $x_1 \uplus y_1 \subseteq x_2 \uplus y_2$ iff $x_1 \subseteq x_2$ and $y_1 \subseteq y_2$.

(80) For all sets x, y, z such that $z \subseteq x \uplus y$ there exist sets x_1, y_1 such that $z = x_1 \uplus y_1$ and $x_1 \subseteq x$ and $y_1 \subseteq y$.

(81) For all sets x_1, y_1, x_2, y_2 holds $x_1 \uplus y_1 = x_2 \uplus y_2$ iff $x_1 = x_2$ and $y_1 = y_2$.

(82) For all sets x_1, y_1, x_2, y_2 holds $(x_1 \uplus y_1) \cup (x_2 \uplus y_2) = x_1 \cup x_2 \uplus y_1 \cup y_2$.

(83) For all sets x_1, y_1, x_2, y_2 holds $(x_1 \uplus y_1) \cap (x_2 \uplus y_2) = x_1 \cap x_2 \uplus y_1 \cap y_2$.

Let C_1, C_2 be coherent spaces. The functor $C_1 \sqcap C_2$ yields a set and is defined by:

(Def.24) $C_1 \sqcap C_2 = \{a \uplus b : a$ ranges over elements of C_1, b ranges over elements of $C_2\}$.

The functor $C_1 \sqcup C_2$ yielding a set is defined as follows:

(Def.25) $C_1 \sqcup C_2 = \{a \uplus \emptyset : a$ ranges over elements of $C_1\} \cup \{\emptyset \uplus b : b$ ranges over elements of $C_2\}$.

The following propositions are true:

(84) Let C_1, C_2 be coherent spaces and let x be a set. Then $x \in C_1 \sqcap C_2$ if and only if there exists an element a of C_1 and there exists an element b of C_2 such that $x = a \uplus b$.

(85) For all coherent spaces C_1, C_2 and for all sets x, y holds $x \uplus y \in C_1 \sqcap C_2$ iff $x \in C_1$ and $y \in C_2$.

(86) For all coherent spaces C_1, C_2 holds $\bigcup(C_1 \sqcap C_2) = \bigcup C_1 \uplus \bigcup C_2$.

(87) For all coherent spaces C_1, C_2 and for all sets x, y holds $x \uplus y \in C_1 \sqcup C_2$ iff $x \in C_1$ and $y = \emptyset$ or $x = \emptyset$ and $y \in C_2$.

(88) Let C_1, C_2 be coherent spaces and let x be a set. Suppose $x \in C_1 \sqcup C_2$. Then there exists an element a of C_1 and there exists an element b of C_2 such that $x = a \uplus b$ but $a = \emptyset$ or $b = \emptyset$.

(89) For all coherent spaces C_1, C_2 holds $\bigcup(C_1 \sqcup C_2) = \bigcup C_1 \uplus \bigcup C_2$.

Let C_1, C_2 be coherent spaces. Observe that $C_1 \sqcap C_2$ is non empty down-closed and binary complete and $C_1 \sqcup C_2$ is non empty down-closed and binary complete.

In the sequel C_1, C_2 will be coherent spaces.

We now state several propositions:

(90) For all sets x, y holds $\langle\langle x, 1\rangle, \langle y, 1\rangle\rangle \in \mathrm{Web}(C_1 \sqcap C_2)$ iff $\langle x, y\rangle \in \mathrm{Web}(C_1)$.

(91) For all sets x, y holds $\langle\langle x, 2\rangle, \langle y, 2\rangle\rangle \in \mathrm{Web}(C_1 \sqcap C_2)$ iff $\langle x, y\rangle \in \mathrm{Web}(C_2)$.

(92) For all sets x, y such that $x \in \bigcup C_1$ and $y \in \bigcup C_2$ holds $\langle\langle x, 1\rangle, \langle y, 2\rangle\rangle \in \mathrm{Web}(C_1 \sqcap C_2)$ and $\langle\langle y, 2\rangle, \langle x, 1\rangle\rangle \in \mathrm{Web}(C_1 \sqcap C_2)$.

(93) For all sets x, y holds $\langle\langle x, 1\rangle, \langle y, 1\rangle\rangle \in \mathrm{Web}(C_1 \sqcup C_2)$ iff $\langle x, y\rangle \in \mathrm{Web}(C_1)$.

(94) For all sets x, y holds $\langle\langle x, 2\rangle, \langle y, 2\rangle\rangle \in \mathrm{Web}(C_1 \sqcup C_2)$ iff $\langle x, y\rangle \in \mathrm{Web}(C_2)$.

(95) For all sets x, y such that $x \in \bigcup C_1$ and $y \in \bigcup C_2$ holds $\langle\langle x, 1\rangle, \langle y, 2\rangle\rangle \notin \mathrm{Web}(C_1 \sqcup C_2)$ and $\langle\langle y, 2\rangle, \langle x, 1\rangle\rangle \notin \mathrm{Web}(C_1 \sqcup C_2)$.

(96) $\neg(C_1 \sqcap C_2) = \neg C_1 \sqcup \neg C_2$.

Let C_1, C_2 be coherent spaces. The functor $C_1 \otimes C_2$ yielding a set is defined as follows:

(Def.26) $C_1 \otimes C_2 = \bigcup\{2^{[: a, b :]}$: a ranges over elements of C_1, b ranges over elements of $C_2\}$.

We now state the proposition

(97) Let C_1, C_2 be coherent spaces and let x be a set. Then $x \in C_1 \otimes C_2$ if and only if there exists an element a of C_1 and there exists an element b of C_2 such that $x \subseteq [: a, b :]$.

Let C_1, C_2 be coherent spaces. One can check that $C_1 \otimes C_2$ is non empty.

Next we state the proposition

(98) For all coherent spaces C_1, C_2 and for every element a of $C_1 \otimes C_2$ holds $\pi_1(a) \in C_1$ and $\pi_2(a) \in C_2$ and $a \subseteq [: \pi_1(a), \pi_2(a) :]$.

Let C_1, C_2 be coherent spaces. One can check that $C_1 \otimes C_2$ is down-closed and binary complete.

Next we state two propositions:

(99) For all coherent spaces C_1, C_2 holds $\bigcup(C_1 \otimes C_2) = [: \bigcup C_1, \bigcup C_2 :]$.

(100) For all sets x_1, y_1, x_2, y_2 holds $\langle\langle x_1, x_2\rangle, \langle y_1, y_2\rangle\rangle \in \mathrm{Web}(C_1 \otimes C_2)$ iff $\langle x_1, y_1\rangle \in \mathrm{Web}(C_1)$ and $\langle x_2, y_2\rangle \in \mathrm{Web}(C_2)$.

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