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On the Decomposition of Finite Sequences

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The notation and terminology used here are introduced in the following papers: [10], [12], [9], [7], [1], [13], [4], [2], [11], [8], [5], [3], and [6].

1. Preliminaries

We introduce degenerated as a synonym of trivial.

Let us observe that every set which is non trivial is also non empty.

In the sequel x, y, z will be arbitrary.

Let us consider x, y. Observe that $\langle x, y \rangle$ is non trivial.

Let us consider x, y, z. Note that $\langle x, y, z \rangle$ is non trivial.

Let f be a non empty finite sequence. One can check that $\mathrm{Rev}(f)$ is non empty.

2. Decomposing a finite sequence

For simplicity we adopt the following rules: f_1, f_2, f_3 will denote finite sequences, p, p_1, p_2, p_3 will be arbitrary, f will denote a finite sequence, and i, k will denote natural numbers.

Next we state a number of propositions:

(3)[1] For every set X and for every i such that $X \subseteq \mathrm{Seg}\, i$ and $1 \in X$ holds $(\mathrm{Sgm}\, X)(1) = 1$.

(4) For every finite sequence f such that $k \in \mathrm{dom}\, f$ and for every i such that $1 \leq i$ and $i < k$ holds $f(i) \neq f(k)$ holds $f(k) \looparrowleft f = k$.

[1]The propositions (1) and (2) have been removed.

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(5) $\langle p_1, p_2\rangle \restriction \mathrm{Seg}\,1 = \langle p_1\rangle$.

(6) $\langle p_1, p_2, p_3\rangle \restriction \mathrm{Seg}\,1 = \langle p_1\rangle$.

(7) $\langle p_1, p_2, p_3\rangle \restriction \mathrm{Seg}\,2 = \langle p_1, p_2\rangle$.

(8) If $p \in \mathrm{rng}\, f_1$, then $p \looparrowleft (f_1 \frown f_2) = p \looparrowleft f_1$.

(9) If $p \in \mathrm{rng}\, f_2 \setminus \mathrm{rng}\, f_1$, then $p \looparrowleft (f_1 \frown f_2) = \mathrm{len}\, f_1 + p \looparrowleft f_2$.

(10) If $p \in \mathrm{rng}\, f_1$, then $f_1 \frown f_2 \rightarrow p = (f_1 \rightarrow p) \frown f_2$.

(11) If $p \in \mathrm{rng}\, f_2 \setminus \mathrm{rng}\, f_1$, then $f_1 \frown f_2 \rightarrow p = f_2 \rightarrow p$.

(12) $f_1 \subseteq f_1 \frown f_2$.

(13) For every set A such that $A \subseteq \mathrm{dom}\, f_1$ holds $(f_1 \frown f_2) \restriction A = f_1 \restriction A$.

(14) If $p \in \mathrm{rng}\, f_1$, then $f_1 \frown f_2 \leftarrow p = f_1 \leftarrow p$.

Let us consider f_1, i. Observe that $f_1 \restriction \mathrm{Seg}\, i$ is finite sequence-like.

The following propositions are true:

(15) If $f_1 \subseteq f_2$, then $f_3 \frown f_1 \subseteq f_3 \frown f_2$.

(16) $(f_1 \frown f_2) \restriction \mathrm{Seg}(\mathrm{len}\, f_1 + i) = f_1 \frown (f_2 \restriction \mathrm{Seg}\, i)$.

(17) If $p \in \mathrm{rng}\, f_2 \setminus \mathrm{rng}\, f_1$, then $f_1 \frown f_2 \leftarrow p = f_1 \frown (f_2 \leftarrow p)$.

(18) For every finite sequence f and for arbitrary p, q such that $p \in \mathrm{rng}\, f$ and $q \in \mathrm{rng}\, f$ and $p \looparrowleft f = q \looparrowleft f$ holds $p = q$.

(19) If $f_1 \frown f_2$ yields p just once, then $p \in \mathrm{rng}\, f_1 \dot{-} \mathrm{rng}\, f_2$.

(20) If $f_1 \frown f_2$ yields p just once and $p \in \mathrm{rng}\, f_1$, then f_1 yields p just once.

(21) If $\mathrm{rng}\, f$ is non trivial, then f is non trivial.

(22) $p \looparrowleft \langle p\rangle = 1$.

(23) $p_1 \looparrowleft \langle p_1, p_2\rangle = 1$.

(24) If $p_1 \neq p_2$, then $p_2 \looparrowleft \langle p_1, p_2\rangle = 2$.

(25) $p_1 \looparrowleft \langle p_1, p_2, p_3\rangle = 1$.

(26) If $p_1 \neq p_2$, then $p_2 \looparrowleft \langle p_1, p_2, p_3\rangle = 2$.

(27) If $p_1 \neq p_3$ and $p_2 \neq p_3$, then $p_3 \looparrowleft \langle p_1, p_2, p_3\rangle = 3$.

(28) For every finite sequence f holds $\mathrm{Rev}(\langle p\rangle \frown f) = (\mathrm{Rev}(f)) \frown \langle p\rangle$.

(29) For every finite sequence f holds $\mathrm{Rev}(\mathrm{Rev}(f)) = f$.

(30) If $x \neq y$, then $\langle x, y\rangle \leftarrow y = \langle x\rangle$.

(31) If $x \neq y$, then $\langle x, y, z\rangle \leftarrow y = \langle x\rangle$.

(32) If $x \neq z$ and $y \neq z$, then $\langle x, y, z\rangle \leftarrow z = \langle x, y\rangle$.

(33) $\langle x, y\rangle \rightarrow x = \langle y\rangle$.

(34) If $x \neq y$, then $\langle x, y, z\rangle \rightarrow y = \langle z\rangle$.

(35) $\langle x, y, z\rangle \rightarrow x = \langle y, z\rangle$.

(36) $\langle z\rangle \rightarrow z = \varepsilon$ and $\langle z\rangle \leftarrow z = \varepsilon$.

(37) If $x \neq y$, then $\langle x, y\rangle \rightarrow y = \varepsilon$.

(38) If $x \neq z$ and $y \neq z$, then $\langle x, y, z\rangle \rightarrow z = \varepsilon$.

(39) If $x \in \mathrm{rng}\, f$ and $y \in \mathrm{rng}(f \leftarrow x)$, then $(f \leftarrow x) \leftarrow y = f \leftarrow y$.

(40) If $x \notin \mathrm{rng}\, f_1$, then $x \looparrowleft (f_1 \frown \langle x\rangle \frown f_2) = \mathrm{len}\, f_1 + 1$.

(41) If f yields x just once, then $x \looparrowleft f + x \looparrowleft \mathrm{Rev}(f) = \mathrm{len}\, f + 1$.

(42) If f yields x just once, then $\mathrm{Rev}(f \leftarrow x) = \mathrm{Rev}(f) \rightarrow x$.

(43) If f yields x just once, then $\mathrm{Rev}(f)$ yields x just once.

3. Finite sequences with elements from a non empty set

We adopt the following convention: D will denote a non empty set, p, p_1, p_2, p_3 will denote elements of D, and f, f_1, f_2 will denote finite sequences of elements of D.

One can prove the following propositions:

(44) If $p \in \mathrm{rng}\, f$, then $f -: p = (f \leftarrow p) \frown \langle p \rangle$.

(45) If $p \in \mathrm{rng}\, f$, then $f :- p = \langle p \rangle \frown (f \rightarrow p)$.

(46) If $f \neq \varepsilon$, then $\pi_1 f \in \mathrm{rng}\, f$.

(47) If $f \neq \varepsilon$, then $(\pi_1 f) \looparrowleft f = 1$.

(48) If $f \neq \varepsilon$ and $\pi_1 f = p$, then $f -: p = \langle p \rangle$ and $f :- p = f$.

(49) $(\langle p_1 \rangle \frown f)_{\restriction 1} = f$.

(50) $\langle p_1, p_2 \rangle_{\restriction 1} = \langle p_2 \rangle$.

(51) $\langle p_1, p_2, p_3 \rangle_{\restriction 1} = \langle p_2, p_3 \rangle$.

(52) If $k \in \mathrm{dom}\, f$ and for every i such that $1 \leq i$ and $i < k$ holds $\pi_i f \neq \pi_k f$, then $(\pi_k f) \looparrowleft f = k$.

(53) If $p_1 \neq p_2$, then $\langle p_1, p_2 \rangle -: p_2 = \langle p_1, p_2 \rangle$.

(54) If $p_1 \neq p_2$, then $\langle p_1, p_2, p_3 \rangle -: p_2 = \langle p_1, p_2 \rangle$.

(55) If $p_1 \neq p_3$ and $p_2 \neq p_3$, then $\langle p_1, p_2, p_3 \rangle -: p_3 = \langle p_1, p_2, p_3 \rangle$.

(56) $\langle p \rangle :- p = \langle p \rangle$ and $\langle p \rangle -: p = \langle p \rangle$.

(57) If $p_1 \neq p_2$, then $\langle p_1, p_2 \rangle :- p_2 = \langle p_2 \rangle$.

(58) If $p_1 \neq p_2$, then $\langle p_1, p_2, p_3 \rangle :- p_2 = \langle p_2, p_3 \rangle$.

(59) If $p_1 \neq p_3$ and $p_2 \neq p_3$, then $\langle p_1, p_2, p_3 \rangle :- p_3 = \langle p_3 \rangle$.

(60) If $x \in \mathrm{rng}\, f$ and $p \in \mathrm{rng}\, f$ and $x \looparrowleft f \leq p \looparrowleft f$, then $x \in \mathrm{rng}(f -: p)$.

(61) If $p \in \mathrm{rng}\, f$ and $p \looparrowleft f > k$, then $p \looparrowleft f = k + p \looparrowleft (f_{\restriction k})$.

(62) If $p \in \mathrm{rng}\, f$ and $p \looparrowleft f > k$, then $p \in \mathrm{rng}(f_{\restriction k})$.

(63) If $k < i$ and $i \in \mathrm{dom}\, f$, then $\pi_i f \in \mathrm{rng}(f_{\restriction k})$.

(64) If $p \in \mathrm{rng}\, f$ and $p \looparrowleft f > k$, then $f_{\restriction k} -: p = (f -: p)_{\restriction k}$.

(65) If $p \in \mathrm{rng}\, f$ and $p \looparrowleft f \neq 1$, then $f_{\restriction 1} -: p = (f -: p)_{\restriction 1}$.

(66) $p \in \mathrm{rng}(f :- p)$.

(67) If $x \in \mathrm{rng}\, f$ and $p \in \mathrm{rng}\, f$ and $x \looparrowleft f \geq p \looparrowleft f$, then $x \in \mathrm{rng}(f :- p)$.

(68) If $p \in \mathrm{rng}\, f$ and $k \leq \mathrm{len}\, f$ and $k \geq p \looparrowleft f$, then $\pi_k f \in \mathrm{rng}(f :- p)$.

(69) If $p \in \mathrm{rng}\, f_1$, then $f_1 \frown f_2 :- p = (f_1 :- p) \frown f_2$.

(70) If $p \in \mathrm{rng}\, f_2 \setminus \mathrm{rng}\, f_1$, then $f_1 \frown f_2 :- p = f_2 :- p$.

(71) If $p \in \mathrm{rng}\, f_1$, then $f_1 \frown f_2 -: p = f_1 -: p$.

(72) If $p \in \operatorname{rng} f_2 \setminus \operatorname{rng} f_1$, then $f_1 \frown f_2 -: p = f_1 \frown (f_2 -: p)$.

(73) $f :- p :- p = f :- p$.

(74) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f \setminus \operatorname{rng}(f -: p_1)$, then $f \rightarrow p_2 = (f \rightarrow p_1) \rightarrow p_2$.

(75) If $p \in \operatorname{rng} f$, then $\operatorname{rng} f = \operatorname{rng}(f -: p) \cup \operatorname{rng}(f :- p)$.

(76) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f \setminus \operatorname{rng}(f -: p_1)$, then $f :- p_1 :- p_2 = f :- p_2$.

(77) If $p \in \operatorname{rng} f$, then $p \looparrowleft (f -: p) = p \looparrowleft f$.

(78) $f \restriction i \restriction i = f \restriction i$.

(79) If $p \in \operatorname{rng} f$, then $f -: p -: p = f -: p$.

(80) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng}(f -: p_1)$, then $f -: p_1 -: p_2 = f -: p_2$.

(81) If $p \in \operatorname{rng} f$, then $(f -: p) \frown ((f :- p)_{\lfloor 1}) = f$.

(82) If $f_1 \neq \varepsilon$, then $(f_1 \frown f_2)_{\lfloor 1} = ((f_1)_{\lfloor 1}) \frown f_2$.

(83) If $p_2 \in \operatorname{rng} f$ and $p_2 \looparrowleft f \neq 1$, then $p_2 \in \operatorname{rng}(f_{\lfloor 1})$.

(84) If $p \in \operatorname{rng} f$, then $p \looparrowleft (f :- p) = 1$.

(86)[2] $(\varepsilon_D)_{\lfloor k} = \varepsilon_D$.

(87) $f_{\lfloor i+k} = (f_{\lfloor i})_{\lfloor k}$.

(88) If $p \in \operatorname{rng} f$ and $p \looparrowleft f > k$, then $f_{\lfloor k} :- p = f :- p$.

(89) If $p \in \operatorname{rng} f$ and $p \looparrowleft f \neq 1$, then $f_{\lfloor 1} :- p = f :- p$.

(90) If $i + k = \operatorname{len} f$, then $\operatorname{Rev}(f_{\lfloor k}) = \operatorname{Rev}(f) \restriction i$.

(91) If $i + k = \operatorname{len} f$, then $\operatorname{Rev}(f \restriction k) = (\operatorname{Rev}(f))_{\lfloor i}$.

(92) If f yields p just once, then $\operatorname{Rev}(f \rightarrow p) = \operatorname{Rev}(f) \leftarrow p$.

(93) If f yields p just once, then $\operatorname{Rev}(f :- p) = \operatorname{Rev}(f) -: p$.

(94) If f yields p just once, then $\operatorname{Rev}(f -: p) = \operatorname{Rev}(f) :- p$.

4. Rotating a finite sequence

Let D be a non empty set. A finite sequence of elements of D is circular if:

(Def.1) $\pi_1 \mathrm{it} = \pi_{\operatorname{len} \mathrm{it}} \mathrm{it}$.

Let us consider D, f, p. The functor $f^{p}_{\circlearrowleft}$ yielding a finite sequence of elements of D is defined by:

(Def.2) (i) $f^{p}_{\circlearrowleft} = (f :- p) \frown ((f -: p)_{\lfloor 1})$ if $p \in \operatorname{rng} f$,

(ii) $f^{p}_{\circlearrowleft} = f$, otherwise.

Let us consider D, let f be a non empty finite sequence of elements of D, and let p be an element of D. One can verify that $f^{p}_{\circlearrowleft}$ is non empty.

Let us consider D. Observe that there exists a finite sequence of elements of D which is circular non empty and trivial and there exists a finite sequence of elements of D which is circular non empty and non trivial.

The following proposition is true

(95) $f^{\pi_1 f}_{\circlearrowleft} = f$.

[2]The proposition (85) has been removed.

Let us consider D, p and let f be a circular non empty finite sequence of elements of D. Observe that $f^{p}_{\circlearrowleft}$ is circular.

We now state a number of propositions:

(96) If f is circular and $p \in \operatorname{rng} f$, then $\operatorname{rng}(f^{p}_{\circlearrowleft}) = \operatorname{rng} f$.

(97) If $p \in \operatorname{rng} f$, then $p \in \operatorname{rng}(f^{p}_{\circlearrowleft})$.

(98) If $p \in \operatorname{rng} f$, then $\pi_1 f^{p}_{\circlearrowleft} = p$.

(99) $(f^{p}_{\circlearrowleft})^{p}_{\circlearrowleft} = f^{p}_{\circlearrowleft}$.

(100) $\langle p \rangle^{p}_{\circlearrowleft} = \langle p \rangle$.

(101) $\langle p_1, p_2 \rangle^{p_1}_{\circlearrowleft} = \langle p_1, p_2 \rangle$.

(102) $\langle p_1, p_2 \rangle^{p_2}_{\circlearrowleft} = \langle p_2, p_2 \rangle$.

(103) $\langle p_1, p_2, p_3 \rangle^{p_1}_{\circlearrowleft} = \langle p_1, p_2, p_3 \rangle$.

(104) If $p_1 \neq p_2$, then $\langle p_1, p_2, p_3 \rangle^{p_2}_{\circlearrowleft} = \langle p_2, p_3, p_2 \rangle$.

(105) If $p_2 \neq p_3$, then $\langle p_1, p_2, p_3 \rangle^{p_3}_{\circlearrowleft} = \langle p_3, p_2, p_3 \rangle$.

(106) For every circular non trivial finite sequence f of elements of D holds $\operatorname{rng}(f_{\lfloor 1}) = \operatorname{rng} f$.

(107) $\operatorname{rng}(f_{\lfloor 1}) \subseteq \operatorname{rng}(f^{p}_{\circlearrowleft})$.

(108) If $p_2 \in \operatorname{rng} f \setminus \operatorname{rng}(f \mathbin{-:} p_1)$, then $(f^{p_1}_{\circlearrowleft})^{p_2}_{\circlearrowleft} = f^{p_2}_{\circlearrowleft}$.

(109) If $p_2 \looparrowleft f \neq 1$ and $p_2 \in \operatorname{rng} f \setminus \operatorname{rng}(f \mathbin{:-} p_1)$, then $(f^{p_1}_{\circlearrowleft})^{p_2}_{\circlearrowleft} = f^{p_2}_{\circlearrowleft}$.

(110) If $p_2 \in \operatorname{rng}(f_{\lfloor 1})$ and f yields p_2 just once, then $(f^{p_1}_{\circlearrowleft})^{p_2}_{\circlearrowleft} = f^{p_2}_{\circlearrowleft}$.

(111) If f is circular and f yields p_2 just once, then $(f^{p_1}_{\circlearrowleft})^{p_2}_{\circlearrowleft} = f^{p_2}_{\circlearrowleft}$.

(112) If f is circular and f yields p just once, then $\operatorname{Rev}(f^{p}_{\circlearrowleft}) = (\operatorname{Rev}(f))^{p}_{\circlearrowleft}$.

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