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On the Lattice of Subgroups of a Group

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The articles [15], [3], [16], [8], [4], [2], [17], [13], [7], [10], [12], [9], [11], [14], [1], [6], and [5] provide the terminology and notation for this paper.

The following propositions are true:

(1) Let G be a group and let H_1, H_2 be subgroups of G. Then the carrier of $H_1 \cap H_2 =$ (the carrier of H_1) $\cap$ (the carrier of H_2).

(2) For every group G and for arbitrary h holds $h \in \mathrm{SubGr}\, G$ iff there exists a strict subgroup H of G such that $h = H$.

(3) Let G be a group, and let A be a subset of the carrier of G, and let H be a strict subgroup of G. If $A =$ the carrier of H, then $\mathrm{gr}(A) = H$.

(4) Let G be a group, and let H_1, H_2 be subgroups of G, and let A be a subset of the carrier of G. If $A =$ (the carrier of H_1) $\cup$ (the carrier of H_2), then $H_1 \sqcup H_2 = \mathrm{gr}(A)$.

(5) Let G be a group, and let H_1, H_2 be subgroups of G, and let g be an element of the carrier of G. If $g \in H_1$ or $g \in H_2$, then $g \in H_1 \sqcup H_2$.

(6) Let G_1, G_2 be groups, and let f be a homomorphism from G_1 to G_2, and let H_1 be a subgroup of G_1. Then there exists a strict subgroup H_2 of G_2 such that the carrier of $H_2 = f^\circ$(the carrier of H_1).

(7) Let G_1, G_2 be groups, and let f be a homomorphism from G_1 to G_2, and let H_2 be a subgroup of G_2. Then there exists a strict subgroup H_1 of G_1 such that the carrier of $H_1 = f^{-1}$ (the carrier of H_2).

(8) Let G_1, G_2 be groups, and let f be a homomorphism from G_1 to G_2, and let H_1, H_2 be subgroups of G_1. Suppose the carrier of $H_1 \subseteq$ the carrier of H_2. Then f°(the carrier of H_1) $\subseteq f^\circ$(the carrier of H_2).

(9) Let G_1, G_2 be groups, and let f be a homomorphism from G_1 to G_2, and let H_1, H_2 be subgroups of G_2. Suppose the carrier of $H_1 \subseteq$ the carrier of H_2. Then f^{-1} (the carrier of H_1) $\subseteq f^{-1}$ (the carrier of H_2).

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(10) Let G_1, G_2 be groups, and let f be a homomorphism from G_1 to G_2, and let H_1, H_2 be subgroups of G_1, and let H_3, H_4 be subgroups of G_2. Suppose the carrier of $H_3 = f^\circ$(the carrier of H_1) and the carrier of $H_4 = f^\circ$(the carrier of H_2). If H_1 is a subgroup of H_2, then H_3 is a subgroup of H_4.

(11) Let G_1, G_2 be groups, and let f be a homomorphism from G_1 to G_2, and let H_1, H_2 be subgroups of G_2, and let H_3, H_4 be subgroups of G_1. Suppose the carrier of $H_3 = f^{-1}$ (the carrier of H_1) and the carrier of $H_4 = f^{-1}$ (the carrier of H_2). If H_1 is a subgroup of H_2, then H_3 is a subgroup of H_4.

(12) Let G_1, G_2 be groups, and let f be a function from the carrier of G_1 into the carrier of G_2, and let A be a subset of the carrier of G_1. Then $f^\circ A \subseteq f^\circ$(the carrier of $\mathrm{gr}(A)$).

(13) Let G_1, G_2 be groups, and let H_1, H_2 be subgroups of G_1, and let f be a function from the carrier of G_1 into the carrier of G_2, and let A be a subset of the carrier of G_1. Suppose $A =$ (the carrier of H_1) $\cup$ (the carrier of H_2). Then f°(the carrier of $H_1 \sqcup H_2$) $= f^\circ$(the carrier of $\mathrm{gr}(A)$).

(14) For every group G and for every subset A of the carrier of G such that $A = \{1_G\}$ holds $\mathrm{gr}(A) = \{\mathbf{1}\}_G$.

(15) For all non empty sets X, Y and for all subsets A_1, A_2 of Y and for every function f from X into Y holds $f^{-1}(A_1 \cup A_2) = f^{-1} A_1 \cup f^{-1} A_2$.

(16) For all non empty sets X, Y and for all subsets A_1, A_2 of X and for every function f from X into Y holds $f^\circ(A_1 \cup A_2) = f^\circ A_1 \cup f^\circ A_2$.

Let G be a group. The functor $\overline{G}$ yields a function from $\mathrm{SubGr}\,G$ into $2^{\text{the carrier of } G}$ and is defined as follows:

(Def.1) For every element h of $\mathrm{SubGr}\,G$ and for every subgroup H of G such that $h = H$ holds $\overline{G}(h) =$ the carrier of H.

Next we state several propositions:

(17) Let G be a group, and let h be an element of $\mathrm{SubGr}\,G$, and let H be a subgroup of G. If $h = H$, then $\overline{G}(h) =$ the carrier of H.

(18) Let G be a group, and let H be a strict subgroup of G, and let x be an element of the carrier of G. Then $x \in \overline{G}(H)$ if and only if $x \in H$.

(19) For every group G and for every strict subgroup H of G holds $1_G \in \overline{G}(H)$.

(20) For every group G and for every strict subgroup H of G holds $\overline{G}(H) \neq \emptyset$.

(21) Let G be a group, and let H be a strict subgroup of G, and let g_1, g_2 be elements of the carrier of G. If $g_1 \in \overline{G}(H)$ and $g_2 \in \overline{G}(H)$, then $g_1 \cdot g_2 \in \overline{G}(H)$.

(22) Let G be a group, and let H be a strict subgroup of G, and let g be an element of the carrier of G. If $g \in \overline{G}(H)$, then $g^{-1} \in \overline{G}(H)$.

(23) For every group G and for all strict subgroups H_1, H_2 of G holds the carrier of $H_1 \cap H_2 = \overline{G}(H_1) \cap \overline{G}(H_2)$.

(24) For every group G and for all strict subgroups H_1, H_2 of G holds $\overline{G}(H_1 \cap H_2) = \overline{G}(H_1) \cap \overline{G}(H_2)$.

Let G be a group and let F be a non empty subset of SubGr G. The functor $\bigcap F$ yielding a strict subgroup of G is defined by:

(Def.2) The carrier of $\bigcap F = \bigcap(\overline{G}^{\circ} F)$.

Next we state several propositions:

(25) For every group G and for every non empty subset F of SubGr G such that $\{\mathbf{1}\}_G \in F$ holds $\bigcap F = \{\mathbf{1}\}_G$.

(26) For every group G and for every element h of SubGr G and for every non empty subset F of SubGr G such that $F = \{h\}$ holds $\bigcap F = h$.

(27) Let G be a group, and let H_1, H_2 be subgroups of G, and let h_1, h_2 be elements of the carrier of $\mathbb{L}_G$. If $h_1 = H_1$ and $h_2 = H_2$, then $h_1 \sqcup h_2 = H_1 \sqcup H_2$.

(28) Let G be a group, and let H_1, H_2 be subgroups of G, and let h_1, h_2 be elements of the carrier of $\mathbb{L}_G$. If $h_1 = H_1$ and $h_2 = H_2$, then $h_1 \sqcap h_2 = H_1 \cap H_2$.

(29) Let G be a group, and let p be an element of the carrier of $\mathbb{L}_G$, and let H be a subgroup of G. If $p = H$, then H is a strict subgroup of G.

(30) Let G be a group, and let H_1, H_2 be subgroups of G, and let p, q be elements of the carrier of $\mathbb{L}_G$. Suppose $p = H_1$ and $q = H_2$. Then $p \sqsubseteq q$ if and only if the carrier of $H_1 \subseteq$ the carrier of H_2.

(31) Let G be a group, and let H_1, H_2 be subgroups of G, and let p, q be elements of the carrier of $\mathbb{L}_G$. If $p = H_1$ and $q = H_2$, then $p \sqsubseteq q$ iff H_1 is a subgroup of H_2.

(32) For every group G holds $\mathbb{L}_G$ is complete.

Let G_1, G_2 be groups and let f be a function from the carrier of G_1 into the carrier of G_2. The functor FuncLatt(f) yielding a function from the carrier of $\mathbb{L}_{(G_1)}$ into the carrier of $\mathbb{L}_{(G_2)}$ is defined by the condition (Def.3).

(Def.3) Let H be a strict subgroup of G_1 and let A be a subset of the carrier of G_2. If $A = f^{\circ}$(the carrier of H), then $(\mathrm{FuncLatt}(f))(H) = \mathrm{gr}(A)$.

One can prove the following propositions:

(33) Let G be a group and let f be a function from the carrier of G into the carrier of G. If $f = \mathrm{id}_{(\text{the carrier of } G)}$, then $\mathrm{FuncLatt}(f) = \mathrm{id}_{(\text{the carrier of } \mathbb{L}_G)}$.

(34) For all groups G_1, G_2 and for every homomorphism f from G_1 to G_2 such that f is one-to-one holds FuncLatt(f) is one-to-one.

(35) For all groups G_1, G_2 and for every homomorphism f from G_1 to G_2 holds $(\mathrm{FuncLatt}(f))(\{\mathbf{1}\}_{(G_1)}) = \{\mathbf{1}\}_{(G_2)}$.

(36) Let G_1, G_2 be groups and let f be a homomorphism from G_1 to G_2. Suppose f is one-to-one. Then FuncLatt(f) is a lower homomorphism between $\mathbb{L}_{(G_1)}$ and $\mathbb{L}_{(G_2)}$.

(37) Let G_1, G_2 be groups and let f be a homomorphism from G_1 to G_2. Then FuncLatt(f) is an upper homomorphism between $\mathbb{L}_{(G_1)}$ and $\mathbb{L}_{(G_2)}$.

(38) Let G_1, G_2 be groups and let f be a homomorphism from G_1 to G_2. If f is one-to-one, then FuncLatt(f) is a homomorphism from $\mathbb{L}_{(G_1)}$ to $\mathbb{L}_{(G_2)}$.

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