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On the Lattice of Subspaces of a Vector Space

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The terminology and notation used here are introduced in the following articles: [18], [11], [5], [17], [6], [20], [14], [15], [13], [1], [16], [10], [19], [3], [4], [2], [12], [9], [7], and [8].

In this paper F denotes a field and V_1 denotes a strict vector space over F.

Let us consider F, V_1. The functor $\mathbb{L}_{(V_1)}$ yields a strict bounded lattice and is defined as follows:

(Def.1) $\mathbb{L}_{(V_1)} = \langle \mathrm{Subspaces}\, V_1, \mathrm{SubJoin}\, V_1, \mathrm{SubMeet}\, V_1 \rangle$.

Let us consider F, V_1. Family of subspaces of V_1 is defined as follows:

(Def.2) For arbitrary x such that $x \in$ it holds x is a subspace of V_1.

Let us consider F, V_1. Note that there exists a family of subspaces of V_1 which is non empty.

Let us consider F, V_1. Then Subspaces V_1 is a non empty family of subspaces of V_1. Let X be a non empty family of subspaces of V_1. We see that the element of X is a subspace of V_1.

Let us consider F, V_1 and let x be an element of Subspaces V_1. The functor $\overline{x}$ yielding a subset of the carrier of V_1 is defined as follows:

(Def.3) There exists a subspace X of V_1 such that $x = X$ and $\overline{x} =$ the carrier of X.

Let us consider F, V_1. The functor $\overline{V_1}$ yielding a function from Subspaces V_1 into $2^{\text{the carrier of } V_1}$ is defined by:

(Def.4) For every element h of Subspaces V_1 and for every subspace H of V_1 such that $h = H$ holds $\overline{V_1}(h) =$ the carrier of H.

We now state two propositions:

(1) For every strict vector space V_1 over F and for every non empty subset H of Subspaces V_1 holds $\overline{V_1}^{\circ} H$ is non empty.

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(2) For every strict vector space V_1 over F and for every strict subspace H of V_1 holds $0_{(V_1)} \in \overline{V_1}(H)$.

Let us consider F, V_1 and let G be a non empty subset of Subspaces V_1. The functor $\bigcap G$ yielding a strict subspace of V_1 is defined by:

(Def.5) The carrier of $\bigcap G = \bigcap(\overline{V_1}^{\circ} G)$.

Next we state several propositions:

(3) Subspaces V_1 = the carrier of $\mathbb{L}_{(V_1)}$.

(4) The meet operation of $\mathbb{L}_{(V_1)} = \mathrm{SubMeet}\, V_1$.

(5) The join operation of $\mathbb{L}_{(V_1)} = \mathrm{SubJoin}\, V_1$.

(6) Let V_1 be a strict vector space over F, and let p, q be elements of the carrier of $\mathbb{L}_{(V_1)}$, and let H_1, H_2 be strict subspaces of V_1. Suppose $p = H_1$ and $q = H_2$. Then $p \sqsubseteq q$ if and only if the carrier of $H_1 \subseteq$ the carrier of H_2.

(7) Let V_1 be a strict vector space over F, and let p, q be elements of the carrier of $\mathbb{L}_{(V_1)}$, and let H_1, H_2 be subspaces of V_1. If $p = H_1$ and $q = H_2$, then $p \sqcup q = H_1 + H_2$.

(8) Let V_1 be a strict vector space over F, and let p, q be elements of the carrier of $\mathbb{L}_{(V_1)}$, and let H_1, H_2 be subspaces of V_1. If $p = H_1$ and $q = H_2$, then $p \sqcap q = H_1 \cap H_2$.

Let us observe that a non empty lattice structure is complete if it satisfies the condition (Def.6).

(Def.6) Let X be a subset of the carrier of it. Then there exists an element a of the carrier of it such that $a \sqsubseteq X$ and for every element b of the carrier of it such that $b \sqsubseteq X$ holds $b \sqsubseteq a$.

The following propositions are true:

(9) For every V_1 holds $\mathbb{L}_{(V_1)}$ is complete.

(10) Let x be arbitrary, and let V_1 be a strict vector space over F, and let S be a subset of the carrier of V_1. If S is non empty and linearly closed, then if $x \in \mathrm{Lin}(S)$, then $x \in S$.

Let F be a field, let A, B be strict vector spaces over F, and let f be a function from the carrier of A into the carrier of B. The functor FuncLatt(f) yields a function from the carrier of $\mathbb{L}_A$ into the carrier of $\mathbb{L}_B$ and is defined by the condition (Def.7).

(Def.7) Let S be a strict subspace of A and let B_0 be a subset of the carrier of B. If $B_0 = f^{\circ}$(the carrier of S), then $(\mathrm{FuncLatt}(f))(S) = \mathrm{Lin}(B_0)$.

Let L_1, L_2 be lattices. A function from the carrier of L_1 into the carrier of L_2 is called a lower homomorphism between L_1 and L_2 if:

(Def.8) For all elements a, b of the carrier of L_1 holds $\mathrm{it}(a \sqcap b) = \mathrm{it}(a) \sqcap \mathrm{it}(b)$.

Let L_1, L_2 be lattices. A function from the carrier of L_1 into the carrier of L_2 is called an upper homomorphism between L_1 and L_2 if:

(Def.9) For all elements a, b of the carrier of L_1 holds $\mathrm{it}(a \sqcup b) = \mathrm{it}(a) \sqcup \mathrm{it}(b)$.

One can prove the following propositions:

(11) Let L_1, L_2 be lattices and let f be a function from the carrier of L_1 into the carrier of L_2. Then f is a homomorphism from L_1 to L_2 if and only if f is an upper homomorphism between L_1 and L_2 and a lower homomorphism between L_1 and L_2.

(12) Let F be a field, and let A, B be strict vector spaces over F, and let f be a function from the carrier of A into the carrier of B. If f is linear, then FuncLatt(f) is an upper homomorphism between $\mathbb{L}_A$ and $\mathbb{L}_B$.

(13) Let F be a field, and let A, B be strict vector spaces over F, and let f be a function from the carrier of A into the carrier of B. Suppose f is one-to-one and linear. Then FuncLatt(f) is a homomorphism from $\mathbb{L}_A$ to $\mathbb{L}_B$.

(14) Let A, B be strict vector spaces over F and let f be a function from the carrier of A into the carrier of B. If f is linear and one-to-one, then FuncLatt(f) is one-to-one.

(15) Let A be a strict vector space over F and let f be a function from the carrier of A into the carrier of A. If $f = \mathrm{id}_{(\text{the carrier of } A)}$, then FuncLatt$(f) = \mathrm{id}_{(\text{the carrier of } \mathbb{L}_A)}$.

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