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Components and Unions of Components

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Summary. First, we generalized **skl** function for a subset of topological spaces the value of which is the component including the set. Second, we introduced a concept of union of components a family of which has good algebraic properties. At the end, we discuss relationship between connectivity of a set as a subset in the whole space and as a subset of a subspace.

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The notation and terminology used in this paper are introduced in the following articles: [8], [11], [3], [1], [10], [5], [9], [7], [2], [6], [12], and [4].

1. The Component of a Subset in a Topological Space

In this paper G_1 will denote a non empty topological space and V, A will denote subsets of the carrier of G_1.

Let G_1 be a non empty topological structure and let V be a subset of the carrier of G_1. The functor Component(V) yields a subset of the carrier of G_1 and is defined by the condition (Def. 1).

(Def. 1) There exists a family F of subsets of G_1 such that for every subset A of the carrier of G_1 holds $A \in F$ iff A is connected and $V \subseteq A$ and $\bigcup F = \mathrm{Component}(V)$.

One can prove the following propositions:

(1) If there exists A such that A is connected and $V \subseteq A$, then $V \subseteq \mathrm{Component}(V)$.

(2) If it is not true that there exists A such that A is connected and $V \subseteq A$, then $\mathrm{Component}(V) = \emptyset$.

(3) $\mathrm{Component}(\emptyset_{(G_1)}) =$ the carrier of G_1.

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(4) For every subset V of the carrier of G_1 such that V is connected holds $\mathrm{Component}(V) \neq \emptyset$.

(5) For every subset V of the carrier of G_1 such that V is connected and $V \neq \emptyset$ holds $\mathrm{Component}(V)$ is connected.

(6) For all subsets V, C of the carrier of G_1 such that V is connected and C is connected holds if $\mathrm{Component}(V) \subseteq C$, then $C = \mathrm{Component}(V)$.

(7) For every subset A of the carrier of G_1 such that A is a component of G_1 holds $\mathrm{Component}(A) = A$.

(8) Let A be a subset of the carrier of G_1. Then A is a component of G_1 if and only if there exists a subset V of the carrier of G_1 such that V is connected and $V \neq \emptyset$ and $A = \mathrm{Component}(V)$.

(9) For every subset V of the carrier of G_1 such that V is connected and $V \neq \emptyset$ holds $\mathrm{Component}(V)$ is a component of G_1.

(10) If A is a component of G_1 and V is connected and $V \subseteq A$ and $V \neq \emptyset$, then $A = \mathrm{Component}(V)$.

(11) For every subset V of the carrier of G_1 such that V is connected and $V \neq \emptyset$ holds $\mathrm{Component}(\mathrm{Component}(V)) = \mathrm{Component}(V)$.

(12) Let A, B be subsets of the carrier of G_1. If A is connected and B is connected and $A \neq \emptyset$ and $A \subseteq B$, then $\mathrm{Component}(A) = \mathrm{Component}(B)$.

(13) For all subsets A, B of the carrier of G_1 such that A is connected and B is connected and $A \neq \emptyset$ and $A \subseteq B$ holds $B \subseteq \mathrm{Component}(A)$.

(14) For all subsets A, B of the carrier of G_1 such that A is connected and $A \cup B$ is connected and $A \neq \emptyset$ holds $A \cup B \subseteq \mathrm{Component}(A)$.

(15) For every subset A of the carrier of G_1 and for every point p of G_1 such that A is connected and $p \in A$ holds $\mathrm{Component}(p) = \mathrm{Component}(A)$.

(16) Let A, B be subsets of the carrier of G_1. Suppose A is connected and B is connected and $A \cap B \neq \emptyset$. Then $A \cup B \subseteq \mathrm{Component}(A)$ and $A \cup B \subseteq \mathrm{Component}(B)$ and $A \subseteq \mathrm{Component}(B)$ and $B \subseteq \mathrm{Component}(A)$.

(17) For every subset A of the carrier of G_1 such that A is connected and $A \neq \emptyset$ holds $\overline{A} \subseteq \mathrm{Component}(A)$.

(18) Let A, B be subsets of the carrier of G_1. Suppose A is a component of G_1 and B is connected and $B \neq \emptyset$ and $A \cap B = \emptyset$. Then $A \cap \mathrm{Component}(B) = \emptyset$.

2. On Unions of Components

Let G_1 be a non empty topological structure. A subset of the carrier of G_1 is called a union of components of G_1 if it satisfies the condition (Def. 2).

(Def. 2) There exists a family F of subsets of G_1 such that for every subset B of the carrier of G_1 such that $B \in F$ holds B is a component of G_1 and it $= \bigcup F$.

The following propositions are true:

(19) $\emptyset_{(G_1)}$ is a union of components of G_1.

(20) Let A be a subset of the carrier of G_1. If $A =$ the carrier of G_1, then A is a union of components of G_1.

(21) Let A be a subset of the carrier of G_1 and let p be a point of G_1. If $p \in A$ and A is a union of components of G_1, then $\mathrm{Component}(p) \subseteq A$.

(22) Let A, B be subsets of the carrier of G_1. Suppose A is a union of components of G_1 and B is a union of components of G_1. Then $A \cup B$ is a union of components of G_1 and $A \cap B$ is a union of components of G_1

(23) Let F_1 be a family of subsets of G_1. Suppose that for every subset A of the carrier of G_1 such that $A \in F_1$ holds A is a union of components of G_1. Then $\bigcup F_1$ is a union of components of G_1.

(24) Let F_1 be a family of subsets of G_1. Suppose that for every subset A of the carrier of G_1 such that $A \in F_1$ holds A is a union of components of G_1. Then $\bigcap F_1$ is a union of components of G_1.

(25) Let A, B be subsets of the carrier of G_1. Suppose A is a union of components of G_1 and B is a union of components of G_1. Then $A \setminus B$ is a union of components of G_1.

3. Operations Down and Up

Let us consider G_1, let B be a subset of the carrier of G_1, and let p be a point of G_1. Let us assume that $p \in B$. The functor $\mathrm{Down}(p, B)$ yielding a point of $G_1 \restriction B$ is defined by:

(Def. 3) $\mathrm{Down}(p, B) = p$.

Let us consider G_1, let B be a subset of the carrier of G_1, and let p be a point of $G_1 \restriction B$. Let us assume that $B \neq \emptyset$. The functor $\mathrm{Up}(p)$ yielding a point of G_1 is defined as follows:

(Def. 4) $\mathrm{Up}(p) = p$.

Let us consider G_1 and let V, B be subsets of the carrier of G_1. Let us assume that $B \neq \emptyset$. The functor $\mathrm{Down}(V, B)$ yields a subset of the carrier of $G_1 \restriction B$ and is defined by:

(Def. 5) $\mathrm{Down}(V, B) = V \cap B$.

Let us consider G_1, let B be a subset of the carrier of G_1, and let V be a subset of the carrier of $G_1 \restriction B$. Let us assume that $B \neq \emptyset$. The functor $\mathrm{Up}(V)$ yielding a subset of the carrier of G_1 is defined as follows:

(Def. 6) $\mathrm{Up}(V) = V$.

Let us consider G_1, let B be a subset of the carrier of G_1, and let p be a point of G_1. Let us assume that $p \in B$. The functor $\mathrm{skl}(p, B)$ yields a subset of the carrier of G_1 and is defined as follows:

(Def. 7) For every point q of $G_1 \restriction B$ such that $q = p$ holds $\mathrm{skl}(p, B) = \mathrm{Component}(q)$.

The following propositions are true:

(26) For every subset B of the carrier of G_1 and for every point p of G_1 such that $p \in B$ holds $\mathrm{skl}(p, B) \neq \emptyset$.

(27) For every subset B of the carrier of G_1 and for every point p of G_1 such that $p \in B$ holds $\mathrm{skl}(p, B) = \mathrm{Component}(\mathrm{Down}(p, B))$.

(28) For all subsets V, B of the carrier of G_1 such that $B \neq \emptyset$ and $V \subseteq B$ holds $\mathrm{Down}(V, B) = V$.

(29) For all subsets V, B of the carrier of G_1 such that $B \neq \emptyset$ and V is open holds $\mathrm{Down}(V, B)$ is open.

(30) For all subsets V, B of the carrier of G_1 such that $B \neq \emptyset$ and $V \subseteq B$ holds $\overline{\mathrm{Down}(V, B)} = \overline{V} \cap B$.

(31) Let B be a subset of the carrier of G_1 and let V be a subset of the carrier of $G_1 \restriction B$.If $B \neq \emptyset$, then $\overline{V} = \overline{\mathrm{Up}(V)} \cap B$.

(32) For all subsets V, B of the carrier of G_1 such that $B \neq \emptyset$ and $V \subseteq B$ holds $\overline{\mathrm{Down}(V, B)} \subseteq \overline{V}$.

(33) Let B be a subset of the carrier of G_1 and let V be a subset of the carrier of $G_1 \restriction B$.If $B \neq \emptyset$ and $V \subseteq B$, then $\mathrm{Down}(\mathrm{Up}(V), B) = V$.

(34) Let V, B be subsets of the carrier of G_1 and let W be a subset of the carrier of $G_1 \restriction B$.If $V = W$ and $V \neq \emptyset$ and $B \neq \emptyset$ and W is connected, then V is connected.

(35) For every subset B of the carrier of G_1 and for every point p of G_1 such that $p \in B$ holds $\mathrm{skl}(p, B)$ is connected.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[3] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[4] Beata Padlewska. Connected spaces. *Formalized Mathematics*, 1(**1**):239–244, 1990.

[5] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[6] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[7] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[8] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[9] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[10] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[11] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[12] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

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