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More on the Lattice of Many Sorted Equivalence Relations

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The notation and terminology used here are introduced in the following papers: [26], [28], [7], [2], [10], [27], [29], [30], [23], [5], [6], [21], [20], [4], [25], [31], [1], [8], [9], [17], [11], [24], [3], [15], [16], [18], [22], [19], [12], [14], and [13].

1. Lattice of Many Sorted Equivalence Relations is Complete

For simplicity we adopt the following convention: I will be a non empty set, M will be a many sorted set indexed by I, x will be arbitrary, and r_1, r_2 will be real numbers.

We now state several propositions:

(1) For every set X holds $x \in$ the carrier of EqRelLatt(X) iff x is an equivalence relation of X.

(2) id_M is an equivalence relation of M.

(3) $[\![M, M]\!]$ is an equivalence relation of M.

(4) $\bot_{\mathrm{EqRelLatt}(M)} = \mathrm{id}_M$.

(5) $\top_{\mathrm{EqRelLatt}(M)} = [\![M, M]\!]$.

Let us consider I, M. Note that EqRelLatt(M) is bounded.

One can prove the following propositions:

(6) Every subset of the carrier of EqRelLatt(M) is a family of many sorted subsets of $[\![M, M]\!]$.

(7) Let a, b be elements of the carrier of EqRelLatt(M) and let A, B be equivalence relations of M. If $a = A$ and $b = B$, then $a \sqsubseteq b$ iff $A \subseteq B$.

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(8) Let X be a subset of the carrier of EqRelLatt(M) and let X_1 be a family of many sorted subsets of $[\![M, M]\!]$. Suppose $X_1 = X$. Let a, b be equivalence relations of M. If $a = \bigcap |{:}X_1{:}|$ and $b \in X$, then $a \subseteq b$.

(9) Let X be a subset of the carrier of EqRelLatt(M) and let X_1 be a family of many sorted subsets of $[\![M, M]\!]$. If $X_1 = X$ and X is non empty, then $\bigcap |{:}X_1{:}|$ is an equivalence relation of M.

Let L be a non empty lattice structure. Let us observe that L is complete if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let X be a subset of the carrier of L. Then there exists an element a of the carrier of L such that $X \sqsubseteq a$ and for every element b of the carrier of L such that $X \sqsubseteq b$ holds $a \sqsubseteq b$.

Next we state the proposition

(10) EqRelLatt(M) is complete.

Let us consider I, M. Observe that EqRelLatt(M) is complete.

We now state the proposition

(11) Let X be a subset of the carrier of EqRelLatt(M) and let X_1 be a family of many sorted subsets of $[\![M, M]\!]$. Suppose $X_1 = X$ and X is non empty. Let a, b be equivalence relations of M. If $a = \bigcap |{:}X_1{:}|$ and $b = \bigsqcap_{\mathrm{EqRelLatt}(M)} X$, then $a = b$.

2. Sublattices inheriting SUP's and INF's

Let L be a lattice and let I_1 be a sublattice of L. We say that I_1 is $\bigsqcap$-inheriting if and only if:

(Def. 2) For every subset X of the carrier of I_1 holds $\bigsqcap_L X \in$ the carrier of I_1.

We say that I_1 is $\bigsqcup$-inheriting if and only if:

(Def. 3) For every subset X of the carrier of I_1 holds $\bigsqcup_L X \in$ the carrier of I_1.

The following propositions are true:

(12) Let L be a lattice, and let L' be a sublattice of L, and let a, b be elements of the carrier of L, and let a', b' be elements of the carrier of L'. If $a = a'$ and $b = b'$, then $a \sqcup b = a' \sqcup b'$ and $a \sqcap b = a' \sqcap b'$.

(13) Let L be a lattice, and let L' be a sublattice of L, and let X be a subset of the carrier of L', and let a be an element of the carrier of L, and let a' be an element of the carrier of L'. If $a = a'$, then $a \sqsubseteq X$ iff $a' \sqsubseteq X$.

(14) Let L be a lattice, and let L' be a sublattice of L, and let X be a subset of the carrier of L', and let a be an element of the carrier of L, and let a' be an element of the carrier of L'. If $a = a'$, then $X \sqsubseteq a$ iff $X \sqsubseteq a'$.

(15) Let L be a complete lattice and let L' be a sublattice of L. If L' is $\bigsqcap$-inheriting, then L' is complete.

(16) Let L be a complete lattice and let L' be a sublattice of L. If L' is $\bigsqcup$-inheriting, then L' is complete.

Let L be a complete lattice. Note that there exists a sublattice of L which is complete.

Let L be a complete lattice. One can verify that every sublattice of L which is $\bigsqcap$-inheriting is also complete and every sublattice of L which is $\bigsqcup$-inheriting is also complete.

Next we state four propositions:

(17) Let L be a complete lattice and let L' be a sublattice of L. Suppose L' is $\bigsqcap$-inheriting. Let A' be a subset of the carrier of L'. Then $\bigsqcap_L A' = \bigsqcap_{L'} A'$.

(18) Let L be a complete lattice and let L' be a sublattice of L. Suppose L' is $\bigsqcup$-inheriting. Let A' be a subset of the carrier of L'. Then $\bigsqcup_L A' = \bigsqcup_{L'} A'$.

(19) Let L be a complete lattice and let L' be a sublattice of L. Suppose L' is $\bigsqcap$-inheriting. Let A be a subset of the carrier of L and let A' be a subset of the carrier of L'. If $A = A'$, then $\bigsqcap A = \bigsqcap A'$.

(20) Let L be a complete lattice and let L' be a sublattice of L. Suppose L' is $\bigsqcup$-inheriting. Let A be a subset of the carrier of L and let A' be a subset of the carrier of L'. If $A = A'$, then $\bigsqcup A = \bigsqcup A'$.

3. Segment of Real Numbers as a Complete Lattice

Let us consider r_1, r_2. Let us assume that $r_1 \leq r_2$. The functor RealSubLatt(r_1, r_2) yields a strict lattice and is defined by the conditions (Def. 4).

(Def. 4) (i) The carrier of RealSubLatt$(r_1, r_2) = [r_1, r_2]$,

(ii) the join operation of RealSubLatt(r_1, r_2) = $\max_{\mathbb{R}} \restriction ([: [r_1, r_2], [r_1, r_2] :]$ **qua** set), and

(iii) the meet operation of RealSubLatt(r_1, r_2) = $\min_{\mathbb{R}} \restriction ([: [r_1, r_2], [r_1, r_2] :]$ **qua** set).

One can prove the following propositions:

(21) For all r_1, r_2 such that $r_1 \leq r_2$ holds RealSubLatt(r_1, r_2) is complete.

(22) There exists sublattice of RealSubLatt$(0, 1)$ which is $\bigsqcup$-inheriting and non $\bigsqcap$-inheriting.

(23) There exists a complete lattice L such that there exists sublattice of L which is $\bigsqcup$-inheriting and non $\bigsqcap$-inheriting.

(24) There exists sublattice of RealSubLatt$(0, 1)$ which is $\bigsqcap$-inheriting and non $\bigsqcup$-inheriting.

(25) There exists a complete lattice L such that there exists sublattice of L which is $\bigsqcap$-inheriting and non $\bigsqcup$-inheriting.

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