FORMALIZED MATHEMATICS
Volume 6, Number 1, 1997
Warsaw University - Białystok

Boolean Posets, Posets under Inclusion and Products of Relational Structures [1]

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Summary. In the paper some notions useful in formalization of [11] are introduced, e.g. the definition of the poset of subsets of a set with inclusion as an ordering relation. Using the theory of many sorted sets authors formulate the definition of product of relational structures.

MML Identifier: YELLOW_1.

The terminology and notation used in this paper are introduced in the following articles: [19], [21], [9], [22], [24], [23], [16], [6], [7], [5], [10], [4], [13], [20], [25], [12], [2], [17], [15], [18], [3], [14], [1], and [8].

1. Boolean Posets and Posets under Inclusion

In this paper X will be a set.

Let L be a lattice. Observe that Poset(L) has l.u.b.'s and g.l.b.'s.

Let L be an upper-bounded lattice. Note that Poset(L) is upper-bounded.

Let L be a lower-bounded lattice. One can check that Poset(L) is lower-bounded.

Let L be a complete lattice. One can verify that Poset(L) is complete.

Let X be a set. Then $\subseteq_X$ is an order in X.

Let X be a set. The functor $\langle X, \subseteq \rangle$ yielding a strict relational structure is defined as follows:

(Def. 1) $\langle X, \subseteq \rangle = \langle X, \subseteq_X \rangle$.

Let X be a set. Observe that $\langle X, \subseteq \rangle$ is reflexive antisymmetric and transitive.

Let X be a non empty set. Observe that $\langle X, \subseteq \rangle$ is non empty.

We now state the proposition

[1]This work was partially supported by Office of Naval Research Grant N00014-95-1-1336.

ISSN 1426–2630

(1) The carrier of $\langle X, \subseteq \rangle = X$ and the internal relation of $\langle X, \subseteq \rangle = {\subseteq}_X$.

Let X be a set. The functor $2^X_\subseteq$ yielding a strict relational structure is defined by:

(Def. 2) $2^X_\subseteq = \mathrm{Poset}(\text{the lattice of subsets of } X)$.

Let X be a set. Note that $2^X_\subseteq$ is non empty reflexive antisymmetric and transitive.

Let X be a set. Note that $2^X_\subseteq$ is complete.

Next we state a number of propositions:

(2) For all elements x, y of $2^X_\subseteq$ holds $x \leq y$ iff $x \subseteq y$.

(3) For every non empty set X and for all elements x, y of $\langle X, \subseteq \rangle$ holds $x \leq y$ iff $x \subseteq y$.

(4) $2^X_\subseteq = \langle 2^X, \subseteq \rangle$.

(5) For every subset Y of 2^X holds $\langle Y, \subseteq \rangle$ is a full relational substructure of $2^X_\subseteq$.

(6) For every non empty set X such that $\langle X, \subseteq \rangle$ has l.u.b.'s and for all elements x, y of $\langle X, \subseteq \rangle$ holds $x \cup y \subseteq x \sqcup y$.

(7) For every non empty set X such that $\langle X, \subseteq \rangle$ has g.l.b.'s and for all elements x, y of $\langle X, \subseteq \rangle$ holds $x \sqcap y \subseteq x \cap y$.

(8) For every non empty set X and for all elements x, y of $\langle X, \subseteq \rangle$ such that $x \cup y \in X$ holds $x \sqcup y = x \cup y$.

(9) For every non empty set X and for all elements x, y of $\langle X, \subseteq \rangle$ such that $x \cap y \in X$ holds $x \sqcap y = x \cap y$.

(10) Let L be a relational structure. Suppose that for all elements x, y of L holds $x \leq y$ iff $x \subseteq y$. Then the internal relation of $L = {\subseteq}_{\text{the carrier of } L}$.

(11) For every non empty set X such that for all sets x, y such that $x \in X$ and $y \in X$ holds $x \cup y \in X$ holds $\langle X, \subseteq \rangle$ has l.u.b.'s.

(12) For every non empty set X such that for all sets x, y such that $x \in X$ and $y \in X$ holds $x \cap y \in X$ holds $\langle X, \subseteq \rangle$ has g.l.b.'s.

(13) For every non empty set X such that $\emptyset \in X$ holds $\bot_{\langle X, \subseteq \rangle} = \emptyset$.

(14) For every non empty set X such that $\bigcup X \in X$ holds $\top_{\langle X, \subseteq \rangle} = \bigcup X$.

(15) For every non empty set X such that $\langle X, \subseteq \rangle$ is upper-bounded holds $\bigcup X \in X$.

(16) For every non empty set X such that $\langle X, \subseteq \rangle$ is lower-bounded holds $\bigcap X \in X$.

(17) For all elements x, y of $2^X_\subseteq$ holds $x \sqcup y = x \cup y$ and $x \sqcap y = x \cap y$.

(18) $\bot_{2^X_\subseteq} = \emptyset$.

(19) $\top_{2^X_\subseteq} = X$.

(20) For every non empty subset Y of $2^X_\subseteq$ holds $\inf Y = \bigcap Y$.

(21) For every subset Y of $2^X_\subseteq$ holds $\sup Y = \bigcup Y$.

(22) For every non empty topological space T and for every subset X of $\langle$the topology of T, $\subseteq\rangle$ holds $\sup X = \bigcup X$.

(23) For every non empty topological space T holds $\bot_{\langle \text{the topology of } T, \subseteq\rangle} = \emptyset$.

(24) For every non empty topological space T holds $\top_{\langle \text{the topology of } T, \subseteq\rangle} =$ the carrier of T.

Let T be a non empty topological space. Observe that $\langle$the topology of T, $\subseteq\rangle$ is complete and non trivial.

We now state the proposition

(25) Let T be a topological space and let F be a family of subsets of T. Then F is open if and only if F is a subset of $\langle$the topology of T, $\subseteq\rangle$.

2. Products of Relational Structures

Let R be a binary relation. We say that R is relational structure yielding if and only if:

(Def. 3) For every set v such that $v \in \operatorname{rng} R$ holds v is a relational structure.

One can check that every function which is relational structure yielding is also 1-sorted yielding.

Let I be a set. One can verify that there exists a many sorted set indexed by I which is relational structure yielding.

Let J be a non empty set, let A be a relational structure yielding many sorted set indexed by J, and let j be an element of J. Then $A(j)$ is a relational structure.

Let I be a set and let J be a relational structure yielding many sorted set indexed by I. The functor $\prod J$ yields a strict relational structure and is defined by the conditions (Def. 4).

(Def. 4) (i) The carrier of $\prod J = \prod \operatorname{support} J$, and

(ii) for all elements x, y of the carrier of $\prod J$ such that $x \in \prod \operatorname{support} J$ holds $x \leq y$ iff there exist functions f, g such that $f = x$ and $g = y$ and for every set i such that $i \in I$ there exists a relational structure R and there exist elements x_1, y_1 of R such that $R = J(i)$ and $x_1 = f(i)$ and $y_1 = g(i)$ and $x_1 \leq y_1$.

Let X be a set and let L be a relational structure. One can verify that $X \longmapsto L$ is relational structure yielding.

Let I be a set and let T be a relational structure. The functor T^I yielding a strict relational structure is defined by:

(Def. 5) $T^I = \prod(I \longmapsto T)$.

Next we state three propositions:

(26) For every relational structure yielding many sorted set J indexed by $\emptyset$ holds $\prod J = \langle\{\emptyset\}, \triangle_{\{\emptyset\}}\rangle$.

(27) For every relational structure Y holds $Y^{\emptyset} = \langle\{\emptyset\}, \triangle_{\{\emptyset\}}\rangle$.

(28) For every set X and for every relational structure Y holds (the carrier of $Y)^X$ = the carrier of Y^X.

Let X be a set and let Y be a non empty relational structure. Note that Y^X is non empty.

Let X be a set and let Y be a reflexive non empty relational structure. Observe that Y^X is reflexive.

Let Y be a non empty relational structure. Observe that $Y^\emptyset$ is trivial.

Let Y be a non empty reflexive relational structure. Note that $Y^\emptyset$ is antisymmetric and has g.l.b.'s and l.u.b.'s.

Let X be a set and let Y be a transitive non empty relational structure. Note that Y^X is transitive.

Let X be a set and let Y be an antisymmetric non empty relational structure. Note that Y^X is antisymmetric.

Let X be a non empty set and let Y be a non empty antisymmetric relational structure with g.l.b.'s. Observe that Y^X has g.l.b.'s.

Let X be a non empty set and let Y be a non empty antisymmetric relational structure with l.u.b.'s. Observe that Y^X has l.u.b.'s.

Let S, T be relational structures. The functor MonMaps(S, T) yielding a strict full relational substructure of $T^{\text{the carrier of } S}$ is defined by the condition (Def. 6).

(Def. 6) Let f be a map from S into T. Then $f \in$ the carrier of MonMaps(S, T) if and only if $f \in$ (the carrier of $T)^{\text{the carrier of } S}$ and f is monotone.

Acknowledgments

The authors would like to express their gratitude to Professor Andrzej Trybulec for his help in formulating mizared definition of the product.

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Received September 20, 1996