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Yoneda Embedding

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The articles [7], [8], [10], [1], [2], [3], [4], [6], [5], and [9] provide the notation and terminology for this paper.

In this paper A is a category, a is an object of A, and f is a morphism of A.

Let us consider A. The functor $\mathrm{EnsHom}\,A$ yields a category and is defined by:

(Def. 1) $\mathrm{EnsHom}\,A = \mathbf{Ens}_{\mathrm{Hom}(A)}$.

Next we state two propositions:

(1) Let f, g be functions and m_1, m_2 be morphisms of $\mathrm{EnsHom}\,A$. If $\mathrm{cod}\,m_1 = \mathrm{dom}\,m_2$ and $\langle\langle \mathrm{dom}\,m_1, \mathrm{cod}\,m_1\rangle, f\rangle = m_1$ and $\langle\langle \mathrm{dom}\,m_2, \mathrm{cod}\,m_2\rangle, g\rangle = m_2$, then $\langle\langle \mathrm{dom}\,m_1, \mathrm{cod}\,m_2\rangle, g \cdot f\rangle = m_2 \cdot m_1$.

(2) $\mathrm{hom}(a, -)$ is a functor from A to $\mathrm{EnsHom}\,A$.

Let us consider A, a. The functor $\mathrm{hom}^{\mathrm{F}}(a, -)$ yields a functor from A to $\mathrm{EnsHom}\,A$ and is defined by:

(Def. 2) $\mathrm{hom}^{\mathrm{F}}(a, -) = \mathrm{hom}(a, -)$.

One can prove the following proposition

(3) For every morphism f of A holds $\mathrm{hom}^{\mathrm{F}}(\mathrm{cod}\,f, -)$ is naturally transformable to $\mathrm{hom}^{\mathrm{F}}(\mathrm{dom}\,f, -)$.

Let us consider A, f. The functor $\mathrm{hom}^{\mathrm{F}}(f, -)$ yields a natural transformation from $\mathrm{hom}^{\mathrm{F}}(\mathrm{cod}\,f, -)$ to $\mathrm{hom}^{\mathrm{F}}(\mathrm{dom}\,f, -)$ and is defined by:

(Def. 3) For every object o of A holds $(\mathrm{hom}^{\mathrm{F}}(f, -))(o) = \langle\langle \mathrm{hom}(\mathrm{cod}\,f, o), \mathrm{hom}(\mathrm{dom}\,f, o)\rangle, \mathrm{hom}(f, \mathrm{id}_o)\rangle$.

Next we state the proposition

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(4) For every element f of the morphisms of A holds $\langle\langle \hom^{\mathrm{F}}(\operatorname{cod} f, -), \hom^{\mathrm{F}}(\operatorname{dom} f, -)\rangle, \hom^{\mathrm{F}}(f, -)\rangle$ is an element of the morphisms of $(\mathrm{EnsHom}\, A)^A$.

Let us consider A. The functor Yoneda A yielding a contravariant functor from A into $(\mathrm{EnsHom}\, A)^A$ is defined by:

(Def. 4) For every morphism f of A holds $(\mathrm{Yoneda}\, A)(f) = \langle\langle \hom^{\mathrm{F}}(\operatorname{cod} f, -), \hom^{\mathrm{F}}(\operatorname{dom} f, -)\rangle, \hom^{\mathrm{F}}(f, -)\rangle$.

Let A, B be categories, let F be a contravariant functor from A into B, and let c be an object of A. The functor $F(c)$ yields an object of B and is defined as follows:

(Def. 5) $F(c) = (\mathrm{Obj}\, F)(c)$.

Next we state the proposition

(5) For every functor F from A to $(\mathrm{EnsHom}\, A)^A$ such that $\mathrm{Obj}\, F$ is one-to-one and F is faithful holds F is one-to-one.

Let C, D be categories and let T be a contravariant functor from C into D. We say that T is faithful if and only if:

(Def. 6) For all objects c, c' of C such that $\hom(c, c') \neq \emptyset$ and for all morphisms f_1, f_2 from c to c' such that $T(f_1) = T(f_2)$ holds $f_1 = f_2$.

The following three propositions are true:

(6) Let F be a contravariant functor from A into $(\mathrm{EnsHom}\, A)^A$. If $\mathrm{Obj}\, F$ is one-to-one and F is faithful, then F is one-to-one.

(7) Yoneda A is faithful.

(8) Yoneda A is one-to-one.

Let C, D be categories and let T be a contravariant functor from C into D. We say that T is full if and only if the condition (Def. 7) is satisfied.

(Def. 7) Let c, c' be objects of C. Suppose $\hom(T(c'), T(c)) \neq \emptyset$. Let g be a morphism from $T(c')$ to $T(c)$. Then $\hom(c, c') \neq \emptyset$ and there exists a morphism f from c to c' such that $g = T(f)$.

The following proposition is true

(9) Yoneda A is full.

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