FORMALIZED MATHEMATICS
Volume 7, Number 1, 1998
University of Białystok

Injective Spaces[1]

Jarosław Gryko
University of Białystok

MML Identifier: WAYBEL18.

The notation and terminology used in this paper have been introduced in the following articles: [20], [16], [13], [1], [14], [7], [6], [5], [17], [10], [11], [12], [19], [15], [8], [22], [18], [2], [3], [9], [21], and [4].

1. Product Topologies

The following propositions are true:

(1) Let x, y, z, Z be sets. Then $Z \subseteq \{x, y, z\}$ if and only if one of the following conditions is satisfied:

(i) $Z = \emptyset$, or
(ii) $Z = \{x\}$, or
(iii) $Z = \{y\}$, or
(iv) $Z = \{z\}$, or
(v) $Z = \{x, y\}$, or
(vi) $Z = \{y, z\}$, or
(vii) $Z = \{x, z\}$, or
(viii) $Z = \{x, y, z\}$.

(2) For every set X and for all families A, B of subsets of X such that $B = A \setminus \{\emptyset\}$ or $A = B \cup \{\emptyset\}$ holds $\mathrm{UniCl}(A) = \mathrm{UniCl}(B)$.

(3) Let T be a topological space and K be a family of subsets of T. Then K is a basis of T if and only if $K \setminus \{\emptyset\}$ is a basis of T.

Let F be a binary relation. We say that F is topological space yielding if and only if:

[1]This work has been supported by KBN Grant 8 T11C 018 12.

ISSN 1426–2630

(Def. 1) For every set x such that $x \in \operatorname{rng} F$ holds x is a topological space.

One can verify that every function which is topological space yielding is also 1-sorted yielding.

Let I be a set. Note that there exists a many sorted set indexed by I which is topological space yielding.

Let I be a set. One can check that there exists a many sorted set indexed by I which is topological space yielding and nonempty.

Let J be a non empty set, let A be a topological space yielding many sorted set indexed by J, and let j be an element of J. Then $A(j)$ is a topological space.

Let I be a set and let J be a topological space yielding many sorted set indexed by I. The product prebasis for J is a family of subsets of $\prod$ (the support of J) and is defined by the condition (Def. 2).

(Def. 2) Let x be a subset of $\prod$ (the support of J). Then $x \in$ the product prebasis for J if and only if there exists a set i and there exists a topological space T and there exists a subset V of T such that $i \in I$ and V is open and $T = J(i)$ and $x = \prod$((the support of J) $+\cdot$ (i, V)).

Next we state the proposition

(4) For every set X and for every family A of subsets of X holds $\langle X$, UniCl(FinMeetCl(A))$\rangle$ is topological space-like.

Let I be a set and let J be a topological space yielding nonempty many sorted set indexed by I. The functor $\prod J$ yielding a strict topological space is defined by:

(Def. 3) The carrier of $\prod J = \prod$ (the support of J) and the product prebasis for J is a prebasis of $\prod J$.

Let I be a set and let J be a topological space yielding nonempty many sorted set indexed by I. One can check that $\prod J$ is non empty.

Let I be a non empty set, let J be a topological space yielding nonempty many sorted set indexed by I, and let i be an element of I. Then $J(i)$ is a non empty topological space.

Let I be a set and let J be a topological space yielding nonempty many sorted set indexed by I. Observe that every element of the carrier of $\prod J$ is function-like and relation-like.

Let I be a non empty set, let J be a topological space yielding nonempty many sorted set indexed by I, let x be an element of the carrier of $\prod J$, and let i be an element of I. Then $x(i)$ is an element of $J(i)$.

Let I be a non empty set, let J be a topological space yielding nonempty many sorted set indexed by I, and let i be an element of I. The functor $\operatorname{proj}(J, i)$ yielding a map from $\prod J$ into $J(i)$ is defined as follows:

(Def. 4) $\operatorname{proj}(J, i) = \operatorname{proj}$(the support of J, i).

One can prove the following propositions:

(5) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, i be an element of I, and P be a subset of the carrier of $J(i)$. Then $(\mathrm{proj}(J,i))^{-1}(P) = \prod((\text{the support of } J) +\cdot (i,P))$.

(6) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, and i be an element of I. Then $\mathrm{proj}(J,i)$ is continuous.

(7) Let X be a non empty topological space, I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, and f be a map from X into $\prod J$. Then f is continuous if and only if for every element i of I holds $\mathrm{proj}(J,i) \cdot f$ is continuous.

2. Injective Spaces

Let Z be a topological structure. We say that Z is injective if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let X be a non empty topological space and f be a map from X into Z. Suppose f is continuous. Let Y be a non empty topological space. Suppose X is a subspace of Y. Then there exists a map g from Y into Z such that g is continuous and $g\restriction$the carrier of $X = f$.

One can prove the following two propositions:

(8) Let I be a non empty set and J be a topological space yielding nonempty many sorted set indexed by I. If for every element i of I holds $J(i)$ is injective, then $\prod J$ is injective.

(9) Let T be a non empty topological space. Suppose T is injective. Let S be a non empty subspace of T. If S is a retract of T, then S is injective.

Let X be a 1-sorted structure, let Y be a topological structure, and let f be a map from X into Y. The functor $\mathrm{Im}\, f$ yielding a subspace of Y is defined as follows:

(Def. 6) $\mathrm{Im}\, f = Y \restriction \mathrm{rng}\, f$.

Let X be a non empty 1-sorted structure, let Y be a non empty topological structure, and let f be a map from X into Y. Note that $\mathrm{Im}\, f$ is non empty.

One can prove the following proposition

(10) Let X be a 1-sorted structure, Y be a topological structure, and f be a map from X into Y. Then the carrier of $\mathrm{Im}\, f = \mathrm{rng}\, f$.

Let X be a 1-sorted structure, let Y be a non empty topological structure, and let f be a map from X into Y. The functor f° yielding a map from X into $\mathrm{Im}\, f$ is defined by:

(Def. 7) $f^\circ = f$.

Next we state the proposition

(11) Let X, Y be non empty topological spaces and f be a map from X into Y. If f is continuous, then f° is continuous.

Let X be a 1-sorted structure, let Y be a non empty topological structure, and let f be a map from X into Y. One can verify that f° is onto.

Let X, Y be topological structures. We say that X is a topological retract of Y if and only if:

(Def. 8) There exists a map f from Y into Y such that f is continuous and $f \cdot f = f$ and $\operatorname{Im} f$ and X are homeomorphic.

The following proposition is true

(12) Let T, S be non empty topological spaces. Suppose T is injective. Let f be a map from T into S. If f° is a homeomorphism, then T is a topological retract of S.

The Sierpiński space is a strict topological structure and is defined by the conditions (Def. 9).

(Def. 9)(i) The carrier of the Sierpiński space $= \{0, 1\}$, and

(ii) the topology of the Sierpiński space $= \{\emptyset, \{1\}, \{0, 1\}\}$.

Let us note that the Sierpiński space is non empty and topological space-like.

One can check that the Sierpiński space is discernible.

Let us note that the Sierpiński space is injective.

Let I be a set and let S be a non empty 1-sorted structure. One can verify that $I \longmapsto S$ is nonempty.

Let I be a set and let T be a topological space. One can check that $I \longmapsto T$ is topological space yielding.

Let I be a set and let L be a reflexive relational structure. One can check that $I \longmapsto L$ is reflexive-yielding.

Let I be a non empty set and let L be a non empty antisymmetric relational structure. Note that $\prod(I \longmapsto L)$ is antisymmetric.

Let I be a non empty set and let L be a non empty transitive relational structure. One can check that $\prod(I \longmapsto L)$ is transitive.

The following two propositions are true:

(13) Let T be a Scott topological augmentation of $2^{1}_{\subseteq}$. Then the topology of $T =$ the topology of the Sierpiński space.

(14) Let I be a non empty set. Then $\{\prod(($the support of $I \longmapsto$ the Sierpiński space$)+\cdot(i, \{1\}))$: i ranges over elements of $I\}$ is a prebasis of $\prod(I \longmapsto$ the Sierpiński space$)$.

Let I be a non empty set and let L be a complete lattice. One can check that $\prod(I \longmapsto L)$ is complete and has l.u.b.'s.

Let I be a non empty set and let X be an algebraic lower-bounded lattice. One can check that $\prod(I \longmapsto X)$ is algebraic.

Next we state several propositions:

(15) Let X be a non empty set. Then there exists a map f from $2_{\subseteq}^{X}$ into $\prod(X \longmapsto 2_{\subseteq}^{1})$ such that f is isomorphic and for every subset Y of X holds $f(Y) = \chi_{Y,X}$.

(16) Let I be a non empty set and T be a Scott topological augmentation of $\prod(I \longmapsto 2_{\subseteq}^{1})$. Then the topology of $T =$ the topology of $\prod(I \longmapsto$ the Sierpiński space).

(17) Let T, S be non empty topological spaces. Suppose the carrier of $T =$ the carrier of S and the topology of $T =$ the topology of S and T is injective. Then S is injective.

(18) For every non empty set I holds every Scott topological augmentation of $\prod I \longmapsto 2_{\subseteq}^{1}$ is injective.

(19) Let T be a T_0-space. Then there exists a non empty set M and there exists a map f from T into $\prod(M \longmapsto$ the Sierpiński space) such that f° is a homeomorphism.

(20) Let T be a T_0-space. Suppose T is injective. Then there exists a non empty set M such that T is a topological retract of $\prod(M \longmapsto$ the Sierpiński space).

References

[1] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[2] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(**1**):93–107, 1997.
[3] Grzegorz Bancerek. The "way-below" relation. *Formalized Mathematics*, 6(**1**):169–176, 1997.
[4] Grzegorz Bancerek. Bases and refinements of topologies. *Formalized Mathematics*, 7(**1**):35–43, 1998.
[5] Grzegorz Bancerek and Andrzej Trybulec. Miscellaneous facts about functions. *Formalized Mathematics*, 5(**4**):485–492, 1996.
[6] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.
[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[8] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[9] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Formalized Mathematics*, 6(**1**):117–121, 1997.
[10] Beata Madras. Product of family of universal algebras. *Formalized Mathematics*, 4(**1**):103–108, 1993.
[11] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Formalized Mathematics*, 5(**2**):167–172, 1996.
[12] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[13] Alexander Yu. Shibakov and Andrzej Trybulec. The Cantor set. *Formalized Mathematics*, 5(**2**):233–236, 1996.
[14] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[16] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[17] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[18] Andrzej Trybulec. Scott topology. *Formalized Mathematics*, 6(**2**):311–319, 1997.
[19] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[20] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[21] Mariusz Żynel and Czesław Byliński. Properties of relational structures, posets, lattices and maps. *Formalized Mathematics*, 6(**1**):123–130, 1997.
[22] Mariusz Żynel and Adam Guzowski. T_0 topological spaces. *Formalized Mathematics*, 5(**1**):75–77, 1996.

Received April 17, 1998