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A Theory of Boolean Valued Functions and Quantifiers with Respect to Partitions

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Summary. In this paper, we define the coordinate of partitions. We also introduce the universal quantifier and the existential quantifier of Boolean valued functions with respect to partitions. Some predicate calculus formulae containing such quantifiers are proved. Such a theory gives a discussion of semantics to usual predicate logic.

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The articles [8], [2], [6], [5], [1], [3], [9], [4], and [7] provide the terminology and notation for this paper.

1. Preliminaries

In this paper Y denotes a non empty set and G denotes a subset of PARTITIONS(Y).

Let X be a set. Then PARTITIONS(X) is a partition family of X.

Let X be a set and let F be a non empty partition family of X. We see that the element of F is a partition of X.

The following proposition is true

(1) Let y be an element of Y. Then there exists a subset X of Y such that

(i) $y \in X$, and

(ii) there exists a function h and there exists a family F of subsets of Y such that $\operatorname{dom} h = G$ and $\operatorname{rng} h = F$ and for every set d such that $d \in G$ holds $h(d) \in d$ and $X = \operatorname{Intersect}(F)$ and $X \neq \emptyset$.

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Let us consider Y and let G be a subset of PARTITIONS(Y). The functor $\bigwedge G$ yielding a partition of Y is defined by the condition (Def. 1).

(Def. 1) Let x be a set. Then $x \in \bigwedge G$ if and only if there exists a function h and there exists a family F of subsets of Y such that $\operatorname{dom} h = G$ and $\operatorname{rng} h = F$ and for every set d such that $d \in G$ holds $h(d) \in d$ and $x = \text{Intersect}(F)$ and $x \neq \emptyset$.

Let us consider Y, let G be a subset of PARTITIONS(Y), and let b be a set. We say that b is upper min depend of G if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) For every partition d of Y such that $d \in G$ holds b is a dependent set of d, and

(ii) for every set e such that $e \subseteq b$ and for every partition d of Y such that $d \in G$ holds e is a dependent set of d holds $e = b$.

One can prove the following proposition

(2) For every element y of Y such that $G \neq \emptyset$ there exists a subset X of Y such that $y \in X$ and X is upper min depend of G.

Let us consider Y and let G be a subset of PARTITIONS(Y). The functor $\bigvee G$ yielding a partition of Y is defined by:

(Def. 3)(i) For every set x holds $x \in \bigvee G$ iff x is upper min depend of G if $G \neq \emptyset$,

(ii) $\bigvee G = \mathcal{I}(Y)$, otherwise.

The following propositions are true:

(3) For every subset G of PARTITIONS(Y) and for every partition P_1 of Y such that $P_1 \in G$ holds $P_1 \Supset \bigwedge G$.

(4) For every subset G of PARTITIONS(Y) and for every partition P_1 of Y such that $P_1 \in G$ holds $P_1 \Subset \bigvee G$.

2. Coordinate and Quantifiers

Let us consider Y and let G be a subset of PARTITIONS(Y). We say that G is a generating family of partitions if and only if:

(Def. 4) $\bigwedge G = \mathcal{I}(Y)$.

Let us consider Y and let G be a subset of PARTITIONS(Y). We say that G is an independent family of partitions if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let h be a function and F be a family of subsets of Y. Suppose $\operatorname{dom} h = G$ and $\operatorname{rng} h = F$ and for every set d such that $d \in G$ holds $h(d) \in d$. Then $\text{Intersect}(F) \neq \emptyset$.

Let us consider Y and let G be a subset of PARTITIONS(Y). We say that G is a coordinate if and only if the conditions (Def. 6) are satisfied.

(Def. 6)(i) G is an independent family of partitions ,

(ii) G is a generating family of partitions, and

(iii) for all partitions d_1, d_2 of Y such that $d_1 \in G$ and $d_2 \in G$ and $d_1 \neq d_2$ holds $d_1 \vee d_2 = \mathcal{O}(Y)$.

Let us consider Y and let P_1 be a partition of Y. Then $\{P_1\}$ is a subset of PARTITIONS(Y).

Let us consider Y, let P_1 be a partition of Y, and let G be a subset of PARTITIONS(Y). The functor CompF(P_1, G) yielding a partition of Y is defined by:

(Def. 7) $\mathrm{CompF}(P_1, G) = \bigwedge G \setminus \{P_1\}$.

Let us consider Y, let a be an element of BVF(Y), let G be a subset of PARTITIONS(Y), and let P_1 be a partition of Y. We say that a is independent of P_1, G if and only if:

(Def. 8) a is dependent of CompF(P_1, G).

Let us consider Y, let a be an element of BVF(Y), let G be a subset of PARTITIONS(Y), and let P_1 be a partition of Y. The functor $\forall_{a,P_1} G$ yielding an element of BVF(Y) is defined by:

(Def. 9) $\forall_{a,P_1} G = \mathrm{INF}(a, \mathrm{CompF}(P_1, G))$.

Let us consider Y, let a be an element of BVF(Y), let G be a subset of PARTITIONS(Y), and let P_1 be a partition of Y. The functor $\exists_{a,P_1} G$ yielding an element of BVF(Y) is defined as follows:

(Def. 10) $\exists_{a,P_1} G = \mathrm{SUP}(a, \mathrm{CompF}(P_1, G))$.

One can prove the following propositions:

(5) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $\forall_{a,P_1} G$ is dependent of CompF(P_1, G).

(6) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $\exists_{a,P_1} G$ is dependent of CompF(P_1, G).

(7) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $\forall_{true(Y),P_1} G = true(Y)$.

(8) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $\exists_{true(Y),P_1} G = true(Y)$.

(9) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $\forall_{false(Y),P_1} G = false(Y)$.

(10) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $\exists_{false(Y),P_1} G =$

$\mathit{false}(Y)$.

(11) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $\forall_{a,P_1} G \Subset a$.

(12) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $a \Subset \exists_{a,P_1} G$.

(13) Let a, b be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $\forall_{a \wedge b,P_1} G = \forall_{a,P_1} G \wedge \forall_{b,P_1} G$.

(14) Let a, b be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $\forall_{a,P_1} G \vee \forall_{b,P_1} G \Subset \forall_{a \vee b,P_1} G$.

(15) Let a, b be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $\forall_{a \Rightarrow b,P_1} G \Subset \forall_{a,P_1} G \Rightarrow \forall_{b,P_1} G$.

(16) Let a, b be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $\exists_{a \vee b,P_1} G = \exists_{a,P_1} G \vee \exists_{b,P_1} G$.

(17) Let a, b be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $\exists_{a \wedge b,P_1} G \Subset \exists_{a,P_1} G \wedge \exists_{b,P_1} G$.

(18) Let a, b be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $\exists_{a,P_1} G \oplus \exists_{b,P_1} G \Subset \exists_{a \oplus b,P_1} G$.

(19) Let a, b be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $\exists_{a,P_1} G \Rightarrow \exists_{b,P_1} G \Subset \exists_{a \Rightarrow b,P_1} G$.

(20) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $\neg\forall_{a,P_1} G = \exists_{\neg a,P_1} G$.

(21) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. If G is a coordinate and $P_1 \in G$, then $\neg\exists_{a,P_1} G = \forall_{\neg a,P_1} G$.

(22) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\forall_{u \Rightarrow a,P_1} G = u \Rightarrow \forall_{a,P_1} G$.

(23) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\forall_{a \Rightarrow u,P_1} G = \exists_{a,P_1} G \Rightarrow u$.

(24) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u

is independent of P_1, G. Then $\forall_{u \vee a, P_1} G = u \vee \forall_{a,P_1} G$.

(25) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\forall_{a \vee u, P_1} G = \forall_{a,P_1} G \vee u$.

(26) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\forall_{a \vee u, P_1} G \Subset \exists_{a,P_1} G \vee u$.

(27) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\forall_{u \wedge a, P_1} G = u \wedge \forall_{a,P_1} G$.

(28) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\forall_{a \wedge u, P_1} G = \forall_{a,P_1} G \wedge u$.

(29) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\forall_{a \wedge u, P_1} G \Subset \exists_{a,P_1} G \wedge u$.

(30) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\forall_{u \oplus a, P_1} G \Subset u \oplus \forall_{a,P_1} G$.

(31) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\forall_{a \oplus u, P_1} G \Subset \forall_{a,P_1} G \oplus u$.

(32) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\forall_{u \Leftrightarrow a, P_1} G \Subset u \Leftrightarrow \forall_{a,P_1} G$.

(33) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\forall_{a \Leftrightarrow u, P_1} G \Subset \forall_{a,P_1} G \Leftrightarrow u$.

(34) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\exists_{u \vee a, P_1} G = u \vee \exists_{a,P_1} G$.

(35) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\exists_{a \vee u, P_1} G = \exists_{a,P_1} G \vee u$.

(36) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\exists_{u \wedge a, P_1} G = u \wedge \exists_{a,P_1} G$.

(37) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\exists_{a \wedge u, P_1} G = \exists_{a,P_1} G \wedge u$.

(38) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $u \Rightarrow \exists_{a,P_1} G \Subset \exists_{u \Rightarrow a,P_1} G$.

(39) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\exists_{a,P_1} G \Rightarrow u \Subset \exists_{a \Rightarrow u,P_1} G$.

(40) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $u \oplus \exists_{a,P_1} G \Subset \exists_{u \oplus a,P_1} G$.

(41) Let a, u be elements of BVF(Y), G be a subset of PARTITIONS(Y), and P_1 be a partition of Y. Suppose G is a coordinate and $P_1 \in G$ and u is independent of P_1, G. Then $\exists_{a,P_1} G \oplus u \Subset \exists_{a \oplus u,P_1} G$.

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