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Introduction to Concept Lattices

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Summary. In this paper we give Mizar formalization of concept lattices. Concept lattices stem from the so called formal concept analysis — a part of applied mathematics that brings mathematical methods into the field of data anylysis and knowledge processing. Our approach follows the one given in [8].

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The papers [3], [14], [4], [5], [1], [15], [12], [10], [13], [11], [2], [7], [9], and [6] provide the notation and terminology for this paper.

1. Formal Contexts

We consider 2-sorted as systems
⟨ objects, a Attributes ⟩,
where the objects constitute a set and the Attributes is a set.

Let C be a 2-sorted. We say that C is empty if and only if:

(Def. 1) The objects of C are empty and the Attributes of C is empty.

We say that C is quasi-empty if and only if:

(Def. 2) The objects of C are empty or the Attributes of C is empty.

Let us note that there exists a 2-sorted which is strict and non empty and there exists a 2-sorted which is strict and non quasi-empty.

One can verify that there exists a 2-sorted which is strict, empty, and quasi-empty.

We consider ContextStr as extensions of 2-sorted as systems
⟨ objects, a Attributes, a Information ⟩,
where the objects constitute a set, the Attributes is a set, and the Information is a relation between the objects and the Attributes.

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One can check that there exists a ContextStr which is strict and non empty and there exists a ContextStr which is strict and non quasi-empty.

A FormalContext is a non quasi-empty ContextStr.

Let C be a 2-sorted.

(Def. 3) An element of the objects of C is said to be an object of C.

(Def. 4) An element of the Attributes of C is said to be a Attribute of C.

Let C be a non quasi-empty 2-sorted. Note that the Attributes of C is non empty and the objects of C is non empty.

Let C be a non quasi-empty 2-sorted. One can check that there exists a subset of the objects of C which is non empty and there exists a subset of the Attributes of C which is non empty.

Let C be a FormalContext, let o be an object of C, and let a be a Attribute of C. We say that o is connected with a if and only if:

(Def. 5) $\langle o, a \rangle \in$ the Information of C.

We introduce o is not connected with a as an antonym of o is connected with a.

2. Derivation Operators

Let C be a FormalContext. The functor ObjectDerivation C yields a function from $2^{\text{the objects of } C}$ into $2^{\text{the Attributes of } C}$ and is defined by the condition (Def. 6).

(Def. 6) Let O be an element of $2^{\text{the objects of } C}$. Then (ObjectDerivation C)(O) $=$ $\{a; a$ ranges over Attribute of C: $\bigwedge_{o\,:\,\text{object of } C} (o \in O \Rightarrow o$ is connected with $a)\}$.

Let C be a FormalContext. The functor AttributeDerivation C yields a function from $2^{\text{the Attributes of } C}$ into $2^{\text{the objects of } C}$ and is defined by the condition (Def. 7).

(Def. 7) Let A be an element of $2^{\text{the Attributes of } C}$. Then (AttributeDerivation C)(A) $=$ $\{o; o$ ranges over objects of C: $\bigwedge_{a\,:\,\text{Attribute of } C} (a \in A \Rightarrow o$ is connected with $a)\}$.

The following propositions are true:

(1) Let C be a FormalContext and o be an object of C. Then (ObjectDerivation C)($\{o\}$) $= \{a; a$ ranges over Attribute of C: o is connected with $a\}$.

(2) Let C be a FormalContext and a be a Attribute of C. Then (AttributeDerivation C)($\{a\}$) $= \{o; o$ ranges over objects of C: o is connected with $a\}$.

(3) For every FormalContext C and for all subsets O_1, O_2 of the objects of C such that $O_1 \subseteq O_2$ holds (ObjectDerivation C)(O_2) $\subseteq$ (ObjectDerivation C)(O_1).

(4) For every FormalContext C and for all subsets A_1, A_2 of the Attributes of C such that $A_1 \subseteq A_2$ holds (AttributeDerivation C)(A_2) $\subseteq$ (AttributeDerivation C)(A_1).

(5) For every FormalContext C and for every subset O of the objects of C holds $O \subseteq$ (AttributeDerivation C)((ObjectDerivation C)(O)).

(6) For every FormalContext C and for every subset A of the Attributes of C holds $A \subseteq$ (ObjectDerivation C)((AttributeDerivation C)(A)).

(7) For every FormalContext C and for every subset O of the objects of C holds (ObjectDerivation C)(O) = (ObjectDerivation C) ((AttributeDerivation C)((ObjectDerivation C)(O))).

(8) For every FormalContext C and for every subset A of the Attributes of C holds (AttributeDerivation C)(A) = (AttributeDerivation C)((ObjectDerivation C)((AttributeDerivation C)(A))).

(9) Let C be a FormalContext, O be a subset of the objects of C, and A be a subset of the Attributes of C. Then $O \subseteq$ (AttributeDerivation C)(A) if and only if $[\!: O, A :\!] \subseteq$ the Information of C.

(10) Let C be a FormalContext, O be a subset of the objects of C, and A be a subset of the Attributes of C. Then $A \subseteq$ (ObjectDerivation C)(O) if and only if $[\!: O, A :\!] \subseteq$ the Information of C.

(11) Let C be a FormalContext, O be a subset of the objects of C, and A be a subset of the Attributes of C. Then $O \subseteq$ (AttributeDerivation C)(A) if and only if $A \subseteq$ (ObjectDerivation C)(O).

Let C be a FormalContext. The functor $\phi(C)$ yielding a map from $2_{\subseteq}^{\text{the objects of } C}$ into $2_{\subseteq}^{\text{the Attributes of } C}$ is defined by:

(Def. 8) $\phi(C)$ = ObjectDerivation C.

Let C be a FormalContext. The functor psi C yields a map from $2_{\subseteq}^{\text{the Attributes of } C}$ into $2_{\subseteq}^{\text{the objects of } C}$ and is defined as follows:

(Def. 9) psi C = AttributeDerivation C.

We now state the proposition

(12) For every FormalContext C holds $\langle \phi(C), \text{psi}\, C \rangle$ is a connection between $2_{\subseteq}^{\text{the objects of } C}$ and $2_{\subseteq}^{\text{the Attributes of } C}$.

Let P, R be non empty relational structures and let C_1 be a connection between P and R. We say that C_1 is co-Galois if and only if the condition (Def. 10) is satisfied.

(Def. 10) There exists a map f from P into R and there exists a map g from R into P such that

(i) $C_1 = \langle f, g \rangle$,
(ii) f is antitone,
(iii) g is antitone, and
(iv) for all elements p_1, p_2 of P and for all elements r_1, r_2 of R holds $p_1 \leqslant g(f(p_1))$ and $r_1 \leqslant f(g(r_1))$.

We now state several propositions:

(13) Let P, R be non empty posets, C_1 be a connection between P and R, f be a map from P into R, and g be a map from R into P. Suppose $C_1 = \langle f, g \rangle$. Then C_1 is co-Galois if and only if for every element p of P and for every element r of R holds $p \leqslant g(r)$ iff $r \leqslant f(p)$.

(14) Let P, R be non empty posets and C_1 be a connection between P and R. Suppose C_1 is co-Galois. Let f be a map from P into R and g be a map from R into P. If $C_1 = \langle f, g \rangle$, then $f = f \cdot (g \cdot f)$ and $g = g \cdot (f \cdot g)$.

(15) For every FormalContext C holds $\langle \phi(C), \mathrm{psi}\, C \rangle$ is co-Galois.

(16) For every FormalContext C and for all subsets O_1, O_2 of the objects of C holds $(\mathrm{ObjectDerivation}\, C)(O_1 \cup O_2) = (\mathrm{ObjectDerivation}\, C)(O_1) \cap (\mathrm{ObjectDerivation}\, C)(O_2)$.

(17) For every FormalContext C and for all subsets A_1, A_2 of the Attributes of C holds $(\mathrm{AttributeDerivation}\, C)(A_1 \cup A_2) = (\mathrm{AttributeDerivation}\, C)(A_1) \cap (\mathrm{AttributeDerivation}\, C)(A_2)$.

(18) For every FormalContext C holds $(\mathrm{ObjectDerivation}\, C)(\emptyset) =$ the Attributes of C.

(19) For every FormalContext C holds $(\mathrm{AttributeDerivation}\, C)(\emptyset) =$ the objects of C.

3. Formal Concepts

Let C be a 2-sorted. We introduce ConceptStr over C which are systems
⟨ a Extent, a Intent ⟩,
where the Extent is a subset of the objects of C and the Intent is a subset of the Attributes of C.

Let C be a 2-sorted and let C_2 be a ConceptStr over C. We say that C_2 is empty if and only if:

(Def. 11) The Extent of C_2 is empty and the Intent of C_2 is empty.

We say that C_2 is quasi-empty if and only if:

(Def. 12) The Extent of C_2 is empty or the Intent of C_2 is empty.

Let C be a non quasi-empty 2-sorted. Observe that there exists a ConceptStr over C which is strict and non empty and there exists a ConceptStr over C which is strict and quasi-empty.

Let C be an empty 2-sorted. Observe that every ConceptStr over C is empty.

Let C be a quasi-empty 2-sorted. Observe that every ConceptStr over C is quasi-empty.

Let C be a FormalContext and let C_2 be a ConceptStr over C. We say that C_2 is concept-like if and only if:

(Def. 13) (ObjectDerivation C)(the Extent of C_2) = the Intent of C_2 and (AttributeDerivation C)(the Intent of C_2) = the Extent of C_2.

Let C be a FormalContext. One can check that there exists a ConceptStr over C which is concept-like and non empty.

Let C be a FormalContext. A FormalConcept of C is a concept-like non empty ConceptStr over C.

Let C be a FormalContext. Note that there exists a FormalConcept of C which is strict.

Next we state four propositions:

(20) Let C be a FormalContext and O be a subset of the objects of C. Then
 (i) $\langle$(AttributeDerivation C)((ObjectDerivation C)(O)), (ObjectDerivation C)(O)$\rangle$ is a FormalConcept of C, and
 (ii) for every subset O' of the objects of C and for every subset A' of the Attributes of C such that $\langle O', A'\rangle$ is a FormalConcept of C and $O \subseteq O'$ holds (AttributeDerivation C)((ObjectDerivation C)(O)) $\subseteq O'$.

(21) Let C be a FormalContext and O be a subset of the objects of C. Then there exists a subset A of the Attributes of C such that $\langle O, A\rangle$ is a FormalConcept of C if and only if (AttributeDerivation C)((ObjectDerivation C)(O)) $= O$.

(22) Let C be a FormalContext and A be a subset of the Attributes of C. Then
 (i) $\langle$(AttributeDerivation C)(A), (ObjectDerivation C)((AttributeDerivation C)(A))$\rangle$ is a FormalConcept of C, and
 (ii) for every subset O' of the objects of C and for every subset A' of the Attributes of C such that $\langle O', A'\rangle$ is a FormalConcept of C and $A \subseteq A'$ holds (ObjectDerivation C)((AttributeDerivation C)(A)) $\subseteq A'$.

(23) Let C be a FormalContext and A be a subset of the Attributes of C. Then there exists a subset O of the objects of C such that $\langle O, A\rangle$ is a FormalConcept of C if and only if (ObjectDerivation C)((AttributeDerivation C)(A)) $= A$.

Let C be a FormalContext and let C_2 be a ConceptStr over C. We say that C_2 is universal if and only if:

(Def. 14) The Extent of C_2 = the objects of C.

Let C be a FormalContext and let C_2 be a ConceptStr over C. We say that C_2 is co-universal if and only if:

(Def. 15) The Intent of C_2 = the Attributes of C.

Let C be a FormalContext. Note that there exists a FormalConcept of C which is strict and universal and there exists a FormalConcept of C which is strict and co-universal.

Let C be a FormalContext. The functor Concept − with − all − Objects C yields a strict universal FormalConcept of C and is defined by the condition (Def. 16).

(Def. 16) There exists a subset O of the objects of C and there exists a subset A of the Attributes of C such that Concept − with − all − Objects $C = \langle O, A\rangle$ and O = (AttributeDerivation C)($\emptyset$) and A = (ObjectDerivation C)((AttributeDerivation C)($\emptyset$)).

Let C be a FormalContext. The functor Concept − with − all − Attributes C yielding a strict co-universal FormalConcept of C is defined by the condition (Def. 17).

(Def. 17) There exists a subset O of the objects of C and there exists a subset A of the Attributes of C such that Concept − with − all − Attributes C = $\langle O, A\rangle$ and O = (AttributeDerivation C)((ObjectDerivation C)($\emptyset$)) and A = (ObjectDerivation C)($\emptyset$).

One can prove the following propositions:

(24) Let C be a FormalContext. Then the Extent of Concept − with − all − Objects C = the objects of C and the Intent of Concept − with − all − Attributes C = the Attributes of C.

(25) Let C be a FormalContext and C_2 be a FormalConcept of C. Then
 (i) if the Extent of $C_2 = \emptyset$, then C_2 is co-universal, and
 (ii) if the Intent of $C_2 = \emptyset$, then C_2 is universal.

(26) Let C be a FormalContext and C_2 be a strict FormalConcept of C. Then
 (i) if the Extent of $C_2 = \emptyset$, then C_2 = Concept − with − all − Attributes C, and
 (ii) if the Intent of $C_2 = \emptyset$, then C_2 = Concept − with − all − Objects C.

(27) Let C be a FormalContext and C_2 be a quasi-empty ConceptStr over C. Suppose C_2 is a FormalConcept of C. Then C_2 is universal or co-universal.

(28) Let C be a FormalContext and C_2 be a quasi-empty ConceptStr over C. If C_2 is a strict FormalConcept of C, then C_2 = Concept − with − all − Objects C or C_2 = Concept − with − all − Attributes C.

Let C be a FormalContext. A non empty set is called a Set of FormalConcepts of C if:

(Def. 18) For every set X such that $X \in$ it holds X is a FormalConcept of C.

Let C be a FormalContext and let F_1 be a Set of FormalConcepts of C. We see that the element of F_1 is a FormalConcept of C.

Let C be a FormalContext and let C_3, C_4 be FormalConcept of C. We say that C_3 is SubConcept of C_4 if and only if:

(Def. 19) The Extent of $C_3 \subseteq$ the Extent of C_4.

We introduce C_4 is SuperConcept of C_3 as a synonym of C_3 is SubConcept of C_4.

One can prove the following propositions:

(29) Let C be a FormalContext and C_3, C_4 be FormalConcept of C. Then C_3 is SuperConcept of C_4 if and only if C_4 is SubConcept of C_3.

(30) Let C be a FormalContext and C_3, C_4 be FormalConcept of C. Then C_3 is SubConcept of C_4 if and only if the Extent of $C_3 \subseteq$ the Extent of C_4.

(31) Let C be a FormalContext and C_3, C_4 be FormalConcept of C. Then C_3 is SubConcept of C_4 if and only if the Intent of $C_4 \subseteq$ the Intent of C_3.

(32) Let C be a FormalContext and C_3, C_4 be FormalConcept of C. Then C_3 is SuperConcept of C_4 if and only if the Extent of $C_4 \subseteq$ the Extent of C_3.

(33) Let C be a FormalContext and C_3, C_4 be FormalConcept of C. Then C_3 is SuperConcept of C_4 if and only if the Intent of $C_3 \subseteq$ the Intent of C_4.

(34) Let C be a FormalContext and C_2 be a FormalConcept of C. Then C_2 is SubConcept of $\mathrm{Concept-with-all-Objects}\,C$ and $\mathrm{Concept-with-all-Attributes}\,C$ is SubConcept of C_2.

4. Concept Lattices

Let C be a FormalContext. The functor $\mathrm{B-carrier}\,C$ yielding a non empty set is defined by the condition (Def. 20).

(Def. 20) $\mathrm{B-carrier}\,C = \{\langle E, I\rangle; E$ ranges over subsets of the objects of C, I ranges over subsets of the Attributes of C: $\langle E, I\rangle$ is non empty $\wedge$ $(\mathrm{ObjectDerivation}\,C)(E) = I$ $\wedge$ $(\mathrm{AttributeDerivation}\,C)(I) = E\}$.

Let C be a FormalContext. Then $\mathrm{B-carrier}\,C$ is a Set of FormalConcepts of C.

Let C be a FormalContext. One can check that $\mathrm{B-carrier}\,C$ is non empty.

One can prove the following proposition

(35) For every FormalContext C and for every set C_2 holds $C_2 \in \mathrm{B-carrier}\,C$ iff C_2 is a strict FormalConcept of C.

Let C be a FormalContext. The functor $\mathrm{B-meet}\,C$ yields a binary operation on $\mathrm{B-carrier}\,C$ and is defined by the condition (Def. 21).

(Def. 21) Let C_3, C_4 be strict FormalConcept of C. Then there exists a subset O of the objects of C and there exists a subset A of the Attributes of C such that

(B − meet C)(C_3, C_4) = $\langle O, A\rangle$ and O = (the Extent of C_3) $\cap$ (the Extent of C_4) and A = (ObjectDerivation C)((AttributeDerivation C)((the Intent of C_3) $\cup$ (the Intent of C_4))).

Let C be a FormalContext. The functor B − join C yielding a binary operation on B − carrier C is defined by the condition (Def. 22).

(Def. 22) Let C_3, C_4 be strict FormalConcept of C. Then there exists a subset O of the objects of C and there exists a subset A of the Attributes of C such that (B − join C)(C_3, C_4) = $\langle O, A\rangle$ and O = (AttributeDerivation C)((ObjectDerivation C)((the Extent of C_3) $\cup$ (the Extent of C_4))) and A = (the Intent of C_3) $\cap$ (the Intent of C_4).

One can prove the following propositions:

(36) For every FormalContext C and for all strict FormalConcept C_3, C_4 of C holds (B − meet C)(C_3, C_4) = (B − meet C)(C_4, C_3).

(37) For every FormalContext C and for all strict FormalConcept C_3, C_4 of C holds (B − join C)(C_3, C_4) = (B − join C)(C_4, C_3).

(38) For every FormalContext C and for all strict FormalConcept C_3, C_4, C_5 of C holds (B − meet C)(C_3, (B − meet C)(C_4, C_5)) = (B − meet C)((B − meet C)(C_3, C_4), C_5).

(39) For every FormalContext C and for all strict FormalConcept C_3, C_4, C_5 of C holds (B − join C)(C_3, (B − join C)(C_4, C_5)) = (B − join C)((B − join C)(C_3, C_4), C_5).

(40) For every FormalContext C and for all strict FormalConcept C_3, C_4 of C holds (B − join C)((B − meet C)(C_3, C_4), C_4) = C_4.

(41) For every FormalContext C and for all strict FormalConcept C_3, C_4 of C holds (B − meet C)(C_3, (B − join C)(C_3, C_4)) = C_3.

(42) For every FormalContext C and for every strict FormalConcept C_2 of C holds (B − meet C)(C_2, Concept − with − all − Objects C) = C_2.

(43) For every FormalContext C and for every strict FormalConcept C_2 of C holds (B − join C)(C_2, Concept − with − all − Objects C) = Concept − with − all − Objects C.

(44) For every FormalContext C and for every strict FormalConcept C_2 of C holds (B − join C)(C_2, Concept − with − all − Attributes C) = C_2.

(45) For every FormalContext C and for every strict FormalConcept C_2 of C holds (B − meet C)(C_2, Concept − with − all − Attributes C) = Concept − with − all − Attributes C.

Let C be a FormalContext. The functor ConceptLattice C yielding a strict non empty lattice structure is defined as follows:

(Def. 23) ConceptLattice C = $\langle$B − carrier C, B − join C, B − meet $C\rangle$.

The following proposition is true

(46) For every FormalContext C holds ConceptLattice C is a lattice.

Let C be a FormalContext. One can verify that ConceptLattice C is strict non empty and lattice-like.

Let C be a FormalContext and let S be a non empty subset of the carrier of ConceptLattice C. We see that the element of S is a FormalConcept of C.

Let C be a FormalContext and let C_2 be an element of the carrier of ConceptLattice C. The functor ${C_2}^{\mathrm{T}}$ yielding a strict FormalConcept of C is defined as follows:

(Def. 24) ${C_2}^{\mathrm{T}} = C_2$.

One can prove the following two propositions:

(47) Let C be a FormalContext and C_3, C_4 be elements of the carrier of ConceptLattice C. Then $C_3 \sqsubseteq C_4$ if and only if ${C_3}^{\mathrm{T}}$ is SubConcept of ${C_4}^{\mathrm{T}}$.

(48) For every FormalContext C holds ConceptLattice C is a complete lattice.

Let C be a FormalContext. Observe that ConceptLattice C is complete.

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