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Graph Theoretical Properties of Arcs in the Plane and Fashoda Meet Theorem

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Summary. We define a graph on an abstract set, edges of which are pairs of any two elements. For any finite sequence of a plane, we give a definition of nodic, which means that edges by a finite sequence are crossed only at terminals. If the first point and the last point of a finite sequence differs, simpleness as a chain and nodic condition imply unfoldedness and s.n.c. condition. We generalize Goboard Theorem, proved by us before, to a continuous case. We call this Fashoda Meet Theorem, which was taken from Fashoda incident of 100 years ago.

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The articles [23], [21], [27], [8], [10], [2], [25], [5], [6], [17], [16], [20], [14], [18], [19], [15], [1], [4], [22], [7], [13], [28], [24], [26], [11], [12], [9], and [3] provide the terminology and notation for this paper.

1. A Graph by Cartesian Product

For simplicity, we adopt the following convention: G denotes a graph, v_1 denotes a finite sequence of elements of the vertices of G, I_1 denotes an oriented chain of G, n, m, k, i, j denote natural numbers, and r, r_1, r_2 denote real numbers.

Next we state four propositions:

(1) $\frac{0}{r} = 0$.

(2) $\sqrt{{r_1}^{\mathbf{2}} + {r_2}^{\mathbf{2}}} \leqslant |r_1| + |r_2|$.

(3) $|r_1| \leqslant \sqrt{{r_1}^{\mathbf{2}} + {r_2}^{\mathbf{2}}}$ and $|r_2| \leqslant \sqrt{{r_1}^{\mathbf{2}} + {r_2}^{\mathbf{2}}}$.

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(4) Let given v_1. Suppose I_1 is Simple and v_1 is oriented vertex seq of I_1. Let given n, m. If $1 \leqslant n$ and $n < m$ and $m \leqslant \operatorname{len} v_1$ and $v_1(n) = v_1(m)$, then $n = 1$ and $m = \operatorname{len} v_1$.

Let X be a set. The functor PGraph X yields a multi graph structure and is defined by:

(Def. 1) $\mathrm{PGraph}\, X = \langle X, [\!:X, X:\!], \pi_1(X \times X), \pi_2(X \times X)\rangle$.

We now state two propositions:

(5) For every non empty set X holds PGraph X is a graph.

(6) For every non empty set X holds the vertices of $\mathrm{PGraph}\, X = X$.

Let f be a finite sequence. The functor PairF f yielding a finite sequence is defined by:

(Def. 2) $\operatorname{len} \mathrm{PairF}\, f = \operatorname{len} f -' 1$ and for every natural number i such that $1 \leqslant i$ and $i < \operatorname{len} f$ holds $(\mathrm{PairF}\, f)(i) = \langle f(i),\ f(i+1)\rangle$.

In the sequel X is a non empty set.

Let X be a non empty set. Then PGraph X is a graph.

The following propositions are true:

(7) Every finite sequence of elements of X is a finite sequence of elements of the vertices of PGraph X.

(8) For every finite sequence f of elements of X holds PairF f is a finite sequence of elements of the edges of PGraph X.

Let X be a non empty set and let f be a finite sequence of elements of X. Then PairF f is a finite sequence of elements of the edges of PGraph X.

We now state two propositions:

(9) Let n be a natural number and f be a finite sequence of elements of X. If $1 \leqslant n$ and $n \leqslant \operatorname{len} \mathrm{PairF}\, f$, then $(\mathrm{PairF}\, f)(n) \in$ the edges of PGraph X.

(10) For every finite sequence f of elements of X holds PairF f is an oriented chain of PGraph X.

Let X be a non empty set and let f be a finite sequence of elements of X. Then PairF f is an oriented chain of PGraph X.

The following proposition is true

(11) Let f be a finite sequence of elements of X and f_1 be a finite sequence of elements of the vertices of PGraph X. If $\operatorname{len} f \geqslant 1$ and $f = f_1$, then f_1 is oriented vertex seq of PairF f.

2. Shortcuts of Finite Sequences in Plane

Let X be a non empty set and let f, g be finite sequences of elements of X. We say that g is Shortcut of f if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $f(1) = g(1)$,

(ii) $f(\operatorname{len} f) = g(\operatorname{len} g)$, and

(iii) there exists a FinSubsequence f_2 of PairF f and there exists a FinSubsequence f_3 of f and there exists an oriented simple chain s_1 of PGraph X and there exists a finite sequence g_1 of elements of the vertices of PGraph X such that Seq $f_2 = s_1$ and Seq $f_3 = g$ and $g_1 = g$ and g_1 is oriented vertex seq of s_1.

We now state four propositions:

(12) For all finite sequences f, g of elements of X such that g is Shortcut of f holds $1 \leqslant \operatorname{len} g$ and $\operatorname{len} g \leqslant \operatorname{len} f$.

(13) Let f be a finite sequence of elements of X. Suppose $\operatorname{len} f \geqslant 1$. Then there exists a finite sequence g of elements of X such that g is Shortcut of f.

(14) For all finite sequences f, g of elements of X such that g is Shortcut of f holds rng PairF $g \subseteq$ rng PairF f.

(15) Let f, g be finite sequences of elements of X. Suppose $f(1) \neq f(\operatorname{len} f)$ and g is Shortcut of f. Then g is one-to-one and rng PairF $g \subseteq$ rng PairF f and $g(1) = f(1)$ and $g(\operatorname{len} g) = f(\operatorname{len} f)$.

Let us consider n and let I_1 be a finite sequence of elements of $\mathcal{E}^n_{\mathrm{T}}$. We say that I_1 is nodic if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let given i, j. Suppose $\mathcal{L}(I_1, i) \cap \mathcal{L}(I_1, j) \neq \emptyset$. Then $\mathcal{L}(I_1, i) \cap \mathcal{L}(I_1, j) = \{I_1(i)\}$ but $I_1(i) = I_1(j)$ or $I_1(i) = I_1(j+1)$ or $\mathcal{L}(I_1, i) \cap \mathcal{L}(I_1, j) = \{I_1(i+1)\}$ but $I_1(i+1) = I_1(j)$ or $I_1(i+1) = I_1(j+1)$ or $\mathcal{L}(I_1, i) = \mathcal{L}(I_1, j)$.

One can prove the following propositions:

(16) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that f is s.n.c. holds f is s.c.c..

(17) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that f is s.c.c. and $\mathcal{L}(f, 1) \cap \mathcal{L}(f, \operatorname{len} f -' 1) = \emptyset$ holds f is s.n.c..

(18) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that f is nodic and PairF f is Simple holds f is s.c.c..

(19) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that f is nodic and PairF f is Simple and $f(1) \neq f(\operatorname{len} f)$ holds f is s.n.c..

(20) For all points p_1, p_2, p_3 of $\mathcal{E}^n_{\mathrm{T}}$ such that there exists a set x such that $x \neq p_2$ and $x \in \mathcal{L}(p_1, p_2) \cap \mathcal{L}(p_2, p_3)$ holds $p_1 \in \mathcal{L}(p_2, p_3)$ or $p_3 \in \mathcal{L}(p_1, p_2)$.

(21) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is s.n.c. and $\mathcal{L}(f, 1) \cap \mathcal{L}(f, 1+1) \subseteq \{\pi_{1+1} f\}$ and $\mathcal{L}(f, \operatorname{len} f -' 2) \cap \mathcal{L}(f, \operatorname{len} f -' 1) \subseteq \{\pi_{\operatorname{len} f -' 1} f\}$. Then f is unfolded.

(22) For every finite sequence f of elements of X such that PairF f is Simple and $f(1) \neq f(\operatorname{len} f)$ holds f is one-to-one and $\operatorname{len} f \neq 1$.

(23) For every finite sequence f of elements of X such that f is one-to-one and $\operatorname{len} f > 1$ holds PairF f is Simple and $f(1) \neq f(\operatorname{len} f)$.

(24) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If f is nodic and PairF f is Simple and $f(1) \neq f(\operatorname{len} f)$, then f is unfolded.

(25) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given i. Suppose g is Shortcut of f and $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} g$. Then there exists a natural number k_1 such that $1 \leqslant k_1$ and $k_1 + 1 \leqslant \operatorname{len} f$ and $\pi_{k_1} f = \pi_i g$ and $\pi_{k_1+1} f = \pi_{i+1} g$ and $f(k_1) = g(i)$ and $f(k_1 + 1) = g(i + 1)$.

(26) For all finite sequences f, g of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that g is Shortcut of f holds $\operatorname{rng} g \subseteq \operatorname{rng} f$.

(27) For all finite sequences f, g of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that g is Shortcut of f holds $\widetilde{\mathcal{L}}(g) \subseteq \widetilde{\mathcal{L}}(f)$.

(28) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. If f is special and g is Shortcut of f, then g is special.

(29) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is special and $2 \leqslant \operatorname{len} f$ and $f(1) \neq f(\operatorname{len} f)$. Then there exists a finite sequence g of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that $2 \leqslant \operatorname{len} g$ and g is special and one-to-one and $\widetilde{\mathcal{L}}(g) \subseteq \widetilde{\mathcal{L}}(f)$ and $f(1) = g(1)$ and $f(\operatorname{len} f) = g(\operatorname{len} g)$ and $\operatorname{rng} g \subseteq \operatorname{rng} f$.

(30) Let f_1, f_4 be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) f_1 is special,

(ii) f_4 is special,

(iii) $2 \leqslant \operatorname{len} f_1$,

(iv) $2 \leqslant \operatorname{len} f_4$,

(v) $f_1(1) \neq f_1(\operatorname{len} f_1)$,

(vi) $f_4(1) \neq f_4(\operatorname{len} f_4)$,

(vii) **X**-coordinate(f_1) lies between (**X**-coordinate(f_1))(1) and (**X**-coordinate(f_1))($\operatorname{len} f_1$),

(viii) **X**-coordinate(f_4) lies between (**X**-coordinate(f_1))(1) and (**X**-coordinate(f_1))($\operatorname{len} f_1$),

(ix) **Y**-coordinate(f_1) lies between (**Y**-coordinate(f_4))(1) and (**Y**-coordinate(f_4))($\operatorname{len} f_4$), and

(x) **Y**-coordinate(f_4) lies between (**Y**-coordinate(f_4))(1) and (**Y**-coordinate(f_4))($\operatorname{len} f_4$).

Then $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_4) \neq \emptyset$.

3. Norm of Points in $\mathcal{E}^n_{\rm T}$

The following proposition is true

(31) For all real numbers a, b, r_1, r_2 such that $a \leqslant r_1$ and $r_1 \leqslant b$ and $a \leqslant r_2$ and $r_2 \leqslant b$ holds $|r_1 - r_2| \leqslant b - a$.

Let us consider n and let p be a point of $\mathcal{E}^n_{\rm T}$. The functor $|p|$ yields a real number and is defined by:

(Def. 5) For every element w of $\mathcal{R}^n$ such that $p = w$ holds $|p| = |w|$.

In the sequel p, p_1, p_2 are points of $\mathcal{E}^n_{\rm T}$.

We now state a number of propositions:

(32) $|0_{\mathcal{E}^n_{\rm T}}| = 0$.

(33) If $|p| = 0$, then $p = 0_{\mathcal{E}^n_{\rm T}}$.

(34) $|p| \geqslant 0$.

(35) $|-p| = |p|$.

(36) $|r \cdot p| = |r| \cdot |p|$.

(37) $|p_1 + p_2| \leqslant |p_1| + |p_2|$.

(38) $|p_1 - p_2| \leqslant |p_1| + |p_2|$.

(39) $|p_1| - |p_2| \leqslant |p_1 + p_2|$.

(40) $|p_1| - |p_2| \leqslant |p_1 - p_2|$.

(41) $|p_1 - p_2| = 0$ iff $p_1 = p_2$.

(42) If $p_1 \neq p_2$, then $|p_1 - p_2| > 0$.

(43) $|p_1 - p_2| = |p_2 - p_1|$.

(44) $|p_1 - p_2| \leqslant |p_1 - p| + |p - p_2|$.

(45) For all points x_1, x_2 of $\mathcal{E}^n$ such that $x_1 = p_1$ and $x_2 = p_2$ holds $|p_1 - p_2| = \rho(x_1, x_2)$.

(46) For every point p of $\mathcal{E}^2_{\rm T}$ holds $|p|^{\mathbf{2}} = |p_{\mathbf{1}}|^{\mathbf{2}} + |p_{\mathbf{2}}|^{\mathbf{2}}$.

(47) For every point p of $\mathcal{E}^2_{\rm T}$ holds $|p| = \sqrt{|p_{\mathbf{1}}|^{\mathbf{2}} + |p_{\mathbf{2}}|^{\mathbf{2}}}$.

(48) For every point p of $\mathcal{E}^2_{\rm T}$ holds $|p| \leqslant |p_{\mathbf{1}}| + |p_{\mathbf{2}}|$.

(49) For all points p_1, p_2 of $\mathcal{E}^2_{\rm T}$ holds $|p_1 - p_2| \leqslant |(p_1)_{\mathbf{1}} - (p_2)_{\mathbf{1}}| + |(p_1)_{\mathbf{2}} - (p_2)_{\mathbf{2}}|$.

(50) For every point p of $\mathcal{E}^2_{\rm T}$ holds $|p_{\mathbf{1}}| \leqslant |p|$ and $|p_{\mathbf{2}}| \leqslant |p|$.

(51) For all points p_1, p_2 of $\mathcal{E}^2_{\rm T}$ holds $|(p_1)_{\mathbf{1}} - (p_2)_{\mathbf{1}}| \leqslant |p_1 - p_2|$ and $|(p_1)_{\mathbf{2}} - (p_2)_{\mathbf{2}}| \leqslant |p_1 - p_2|$.

(52) If $p \in \mathcal{L}(p_1, p_2)$, then there exists r such that $0 \leqslant r$ and $r \leqslant 1$ and $p = (1 - r) \cdot p_1 + r \cdot p_2$.

(53) If $p \in \mathcal{L}(p_1, p_2)$, then $|p - p_1| \leqslant |p_1 - p_2|$ and $|p - p_2| \leqslant |p_1 - p_2|$.

4. Extended Goboard Theorem and Fashoda Meet Theorem

In the sequel M denotes a metric space.

Next we state several propositions:

(54) For all subsets P, Q of $M_{\rm top}$ such that $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact holds $\mathrm{dist}^{\min}_{\min}(P, Q) \geqslant 0$.

(55) Let P, Q be subsets of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then $P \cap Q = \emptyset$ if and only if $\mathrm{dist}^{\min}_{\min}(P, Q) > 0$.

(56) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and a, c, d be real numbers. Suppose that

(i) $1 \leqslant \mathrm{len}\, f$,

(ii) **X**-coordinate(f) lies between (**X**-coordinate(f))(1) and (**X**-coordinate(f))($\mathrm{len}\, f$),

(iii) **Y**-coordinate(f) lies between c and d,

(iv) $a > 0$, and

(v) for every i such that $1 \leqslant i$ and $i+1 \leqslant \mathrm{len}\, f$ holds $|\pi_i f - \pi_{i+1} f| < a$.

Then there exists a finite sequence g of elements of $\mathcal{E}^2_{\rm T}$ such that

(vi) g is special,

(vii) $g(1) = f(1)$,

(viii) $g(\mathrm{len}\, g) = f(\mathrm{len}\, f)$,

(ix) $\mathrm{len}\, g \geqslant \mathrm{len}\, f$,

(x) **X**-coordinate(g) lies between (**X**-coordinate(f))(1) and (**X**-coordinate(f))($\mathrm{len}\, f$),

(xi) **Y**-coordinate(g) lies between c and d,

(xii) for every j such that $j \in \mathrm{dom}\, g$ there exists k such that $k \in \mathrm{dom}\, f$ and $|\pi_j g - \pi_k f| < a$, and

(xiii) for every j such that $1 \leqslant j$ and $j+1 \leqslant \mathrm{len}\, g$ holds $|\pi_j g - \pi_{j+1} g| < a$.

(57) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and a, c, d be real numbers. Suppose that

(i) $1 \leqslant \mathrm{len}\, f$,

(ii) **Y**-coordinate(f) lies between (**Y**-coordinate(f))(1) and (**Y**-coordinate(f))($\mathrm{len}\, f$),

(iii) **X**-coordinate(f) lies between c and d,

(iv) $a > 0$, and

(v) for every i such that $1 \leqslant i$ and $i+1 \leqslant \mathrm{len}\, f$ holds $|\pi_i f - \pi_{i+1} f| < a$.

Then there exists a finite sequence g of elements of $\mathcal{E}^2_{\rm T}$ such that

(vi) g is special,

(vii) $g(1) = f(1)$,

(viii) $g(\mathrm{len}\, g) = f(\mathrm{len}\, f)$,

(ix) $\mathrm{len}\, g \geqslant \mathrm{len}\, f$,

(x) $\mathbf{Y}$-coordinate(g) lies between ($\mathbf{Y}$-coordinate(f))(1) and ($\mathbf{Y}$-coordinate(f))($\operatorname{len} f$),

(xi) $\mathbf{X}$-coordinate(g) lies between c and d,

(xii) for every j such that $j \in \operatorname{dom} g$ there exists k such that $k \in \operatorname{dom} f$ and $|\pi_j g - \pi_k f| < a$, and

(xiii) for every j such that $1 \leqslant j$ and $j + 1 \leqslant \operatorname{len} g$ holds $|\pi_j g - \pi_{j+1} g| < a$.

(58) For every subset P of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that P is an arc from p_1 to p_2 holds $p_1 \neq p_2$.

(59) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that $1 \leqslant \operatorname{len} f$ holds $\operatorname{len} \mathbf{X}$-coordinate($f$) $= \operatorname{len} f$ and ($\mathbf{X}$-coordinate(f))(1) $= (\pi_1 f)_{\mathbf{1}}$ and ($\mathbf{X}$-coordinate(f))($\operatorname{len} f$) $= (\pi_{\operatorname{len} f} f)_{\mathbf{1}}$.

(60) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that $1 \leqslant \operatorname{len} f$ holds $\operatorname{len} \mathbf{Y}$-coordinate($f$) $= \operatorname{len} f$ and ($\mathbf{Y}$-coordinate(f))(1) $= (\pi_1 f)_{\mathbf{2}}$ and ($\mathbf{Y}$-coordinate(f))($\operatorname{len} f$) $= (\pi_{\operatorname{len} f} f)_{\mathbf{2}}$.

(61) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ and for every i such that $i \in \operatorname{dom} f$ holds ($\mathbf{X}$-coordinate(f))(i) $= (\pi_i f)_{\mathbf{1}}$ and ($\mathbf{Y}$-coordinate(f))(i) $= (\pi_i f)_{\mathbf{2}}$.

(62) Let P, Q be non empty subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) P is an arc from p_1 to p_2,

(ii) Q is an arc from q_1 to q_2,

(iii) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in P$ holds $(p_1)_{\mathbf{1}} \leqslant p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leqslant (p_2)_{\mathbf{1}}$,

(iv) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in Q$ holds $(p_1)_{\mathbf{1}} \leqslant p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leqslant (p_2)_{\mathbf{1}}$,

(v) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in P$ holds $(q_1)_{\mathbf{2}} \leqslant p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leqslant (q_2)_{\mathbf{2}}$, and

(vi) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in Q$ holds $(q_1)_{\mathbf{2}} \leqslant p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leqslant (q_2)_{\mathbf{2}}$.

Then $P \cap Q \neq \emptyset$.

In the sequel X, Y are non empty topological spaces.

We now state three propositions:

(63) Let f be a map from X into Y, P be a non empty subset of the carrier of Y, and f_1 be a map from X into $Y \restriction P$. If $f = f_1$ and f is continuous, then f_1 is continuous.

(64) Let f be a map from X into Y and P be a non empty subset of the carrier of Y. Suppose X is compact and Y is a T_2 space and f is continuous and one-to-one and $P = \operatorname{rng} f$. Then there exists a map f_1 from X into $Y \restriction P$ such that $f = f_1$ and f_1 is a homeomorphism.

(65) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and O, I be points of $\mathbb{I}$. Suppose that

(i) $O = 0$,

(ii) $I = 1$,

(iii) f is continuous and one-to-one,
(iv) g is continuous and one-to-one,
(v) $f(O)_{\mathbf{1}} = a$,
(vi) $f(I)_{\mathbf{1}} = b$,
(vii) $g(O)_{\mathbf{2}} = c$,
(viii) $g(I)_{\mathbf{2}} = d$, and
(ix) for every point r of $\mathbb{I}$ holds $a \leqslant f(r)_{\mathbf{1}}$ and $f(r)_{\mathbf{1}} \leqslant b$ and $a \leqslant g(r)_{\mathbf{1}}$ and $g(r)_{\mathbf{1}} \leqslant b$ and $c \leqslant f(r)_{\mathbf{2}}$ and $f(r)_{\mathbf{2}} \leqslant d$ and $c \leqslant g(r)_{\mathbf{2}}$ and $g(r)_{\mathbf{2}} \leqslant d$.

Then $\mathrm{rng}\, f \cap \mathrm{rng}\, g \neq \emptyset$.

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