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A Theory of Partitions. Part I

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Summary. In this paper, we define join and meet operations between partitions. The properties of these operations are proved. Then we introduce the correspondence between partitions and equivalence relations which preserve join and meet operations. The properties of these relationships are proved.

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The notation and terminology used in this paper have been introduced in the following articles: [9], [6], [5], [2], [3], [1], [10], [4], [8], and [7].

1. Preliminaries

For simplicity, we use the following convention: Y is a non empty set, P_1, P_2 are partitions of Y, A, B are subsets of Y, i is a natural number, x, y, x_1, x_2, z_0 are sets, and X, V, d, t, S_1, S_2 are sets.

The following proposition is true

(1) If $X \in P_1$ and $V \in P_1$ and $X \subseteq V$, then $X = V$.

Let us consider S_1, S_2. We introduce $S_1 \Subset S_2$ and $S_2 \Supset S_1$ as synonyms of S_1 is finer than S_2.

We now state several propositions:

(2) For every partition P_1 of Y holds $P_1 \Supset P_1$.

(3) $\bigcup(S_1 \setminus \{\emptyset\}) = \bigcup S_1$.

(4) For all partitions P_1, P_2 of Y such that $P_1 \Supset P_2$ and $P_2 \Supset P_1$ holds $P_2 \subseteq P_1$.

(5) For all partitions P_1, P_2 of Y such that $P_1 \Supset P_2$ and $P_2 \Supset P_1$ holds $P_1 = P_2$.

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(7)[1] For all partitions P_1, P_2 of Y such that $P_1 \Supset P_2$ holds P_1 is coarser than P_2.

Let us consider Y, let P_1 be a partition of Y, and let b be a set. We say that b is a dependent set of P_1 if and only if:

(Def. 1) There exists a set B such that $B \subseteq P_1$ and $B \neq \emptyset$ and $b = \bigcup B$.

Let us consider Y, let P_1, P_2 be partitions of Y, and let b be a set. We say that b is a minimal dependent set of P_1 and P_2 if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) b is a dependent set of P_1 and a dependent set of P_2, and

(ii) for every set d such that $d \subseteq b$ and d is a dependent set of P_1 and a dependent set of P_2 holds $d = b$.

We now state several propositions:

(8) For all partitions P_1, P_2 of Y such that $P_1 \Supset P_2$ and for every set b such that $b \in P_1$ holds b is a dependent set of P_2.

(9) For every partition P_1 of Y holds Y is a dependent set of P_1.

(10) Let F be a family of subsets of Y. Suppose $\mathrm{Intersect}(F) \neq \emptyset$ and for every X such that $X \in F$ holds X is a dependent set of P_1. Then $\mathrm{Intersect}(F)$ is a dependent set of P_1.

(11) Let X_0, X_1 be subsets of Y. Suppose X_0 is a dependent set of P_1 and X_1 is a dependent set of P_1 and X_0 meets X_1. Then $X_0 \cap X_1$ is a dependent set of P_1.

(12) For every subset X of Y such that X is a dependent set of P_1 and $X \neq Y$ holds X^{c} is a dependent set of P_1.

(13) For every element y of Y there exists a subset X of Y such that $y \in X$ and X is a minimal dependent set of P_1 and P_2.

(14) For every partition P of Y and for every element y of Y there exists a subset A of Y such that $y \in A$ and $A \in P$.

Let Y be a non empty set. One can verify that every partition of Y is non empty.

Let Y be a set. The functor $\mathrm{PARTITIONS}(Y)$ is defined by:

(Def. 3) For every set x holds $x \in \mathrm{PARTITIONS}(Y)$ iff x is a partition of Y.

Let Y be a set. One can check that $\mathrm{PARTITIONS}(Y)$ is non empty.

2. Join and Meet Operation Between Partitions

Let us consider Y and let P_1, P_2 be partitions of Y. The functor $P_1 \wedge P_2$ yielding a partition of Y is defined by:

[1]The proposition (6) has been removed.

(Def. 4) $P_1 \wedge P_2 = P_1 \Cap P_2 \setminus \{\emptyset\}$.

Let us observe that the functor $P_1 \wedge P_2$ is commutative.

One can prove the following propositions:

(15) For every partition P_1 of Y holds $P_1 \wedge P_1 = P_1$.

(16) For all partitions P_1, P_2, P_3 of Y holds $P_1 \wedge P_2 \wedge P_3 = P_1 \wedge P_2 \wedge P_3$.

(17) For all partitions P_1, P_2 of Y holds $P_1 \Supset P_1 \wedge P_2$.

(18) For all partitions P_1, P_2, P_3 of Y such that $P_1 \Supset P_2$ and $P_2 \Supset P_3$ holds $P_1 \Supset P_3$.

Let us consider Y and let P_1, P_2 be partitions of Y. The functor $P_1 \vee P_2$ yielding a partition of Y is defined by:

(Def. 5) For every d holds $d \in P_1 \vee P_2$ iff d is a minimal dependent set of P_1 and P_2.

Let us observe that the functor $P_1 \vee P_2$ is commutative.

One can prove the following propositions:

(19) For all partitions P_1, P_2 of Y holds $P_1 \Subset P_1 \vee P_2$.

(20) For every partition P_1 of Y holds $P_1 \vee P_1 = P_1$.

(21) For all partitions P_1, P_3 of Y such that $P_1 \Subset P_3$ and $x \in P_3$ and $z_0 \in P_1$ and $t \in x$ and $t \in z_0$ holds $z_0 \subseteq x$.

(22) For all partitions P_1, P_2 of Y such that $x \in P_1 \vee P_2$ and $z_0 \in P_1$ and $t \in x$ and $t \in z_0$ holds $z_0 \subseteq x$.

3. Partitions and Equivalence Relations

We now state the proposition

(23) Let P_1 be a partition of Y. Then there exists an equivalence relation R_1 of Y such that for all x, y holds $\langle x, y \rangle \in R_1$ if and only if the following conditions are satisfied:

(i) $x \in Y$,

(ii) $y \in Y$, and

(iii) there exists A such that $A \in P_1$ and $x \in A$ and $y \in A$.

Let us consider Y. The functor $\mathrm{Rel}(Y)$ yields a function and is defined by the conditions (Def. 6).

(Def. 6)(i) $\mathrm{dom}\,\mathrm{Rel}(Y) = \mathrm{PARTITIONS}(Y)$, and

(ii) for every x such that $x \in \mathrm{PARTITIONS}(Y)$ there exists an equivalence relation R_1 of Y such that $(\mathrm{Rel}(Y))(x) = R_1$ and for all sets x_1, x_2 holds $\langle x_1, x_2 \rangle \in R_1$ iff $x_1 \in Y$ and $x_2 \in Y$ and there exists A such that $A \in x$ and $x_1 \in A$ and $x_2 \in A$.

Let Y be a non empty set and let P_1 be a partition of Y. The functor $\equiv_{(P_1)}$ yielding an equivalence relation of Y is defined as follows:

(Def. 7) $\equiv_{(P_1)} = (\mathrm{Rel}(Y))(P_1)$.

The following propositions are true:

(24) For all partitions P_1, P_2 of Y holds $P_1 \Subset P_2$ iff $\equiv_{(P_1)} \subseteq \equiv_{(P_2)}$.

(25) Let P_1, P_2 be partitions of Y, p_0, x, y be sets, and f be a finite sequence of elements of Y. Suppose that

(i) $p_0 \subseteq Y$,

(ii) $x \in p_0$,

(iii) $f(1) = x$,

(iv) $f(\mathrm{len}\, f) = y$,

(v) $1 \leqslant \mathrm{len}\, f$,

(vi) for every i such that $1 \leqslant i$ and $i < \mathrm{len}\, f$ there exist sets p_2, p_3, u such that $p_2 \in P_1$ and $p_3 \in P_2$ and $f(i) \in p_2$ and $u \in p_2$ and $u \in p_3$ and $f(i+1) \in p_3$, and

(vii) p_0 is a dependent set of P_1 and a dependent set of P_2.

Then $y \in p_0$.

(26) Let R_2, R_3 be equivalence relations of Y, f be a finite sequence of elements of Y, and x, y be sets. Suppose that

(i) $x \in Y$,

(ii) $y \in Y$,

(iii) $f(1) = x$,

(iv) $f(\mathrm{len}\, f) = y$,

(v) $1 \leqslant \mathrm{len}\, f$, and

(vi) for every i such that $1 \leqslant i$ and $i < \mathrm{len}\, f$ there exists a set u such that $u \in Y$ and $\langle f(i), u \rangle \in R_2 \cup R_3$ and $\langle u, f(i+1) \rangle \in R_2 \cup R_3$.

Then $\langle x, y \rangle \in R_2 \sqcup R_3$.

(27) For all partitions P_1, P_2 of Y holds $\equiv_{P_1 \vee P_2} = \equiv_{(P_1)} \sqcup \equiv_{(P_2)}$.

(28) For all partitions P_1, P_2 of Y holds $\equiv_{P_1 \wedge P_2} = \equiv_{(P_1)} \cap \equiv_{(P_2)}$.

(29) For all partitions P_1, P_2 of Y such that $\equiv_{(P_1)} = \equiv_{(P_2)}$ holds $P_1 = P_2$.

(30) For all partitions P_1, P_2, P_3 of Y holds $P_1 \vee P_2 \vee P_3 = P_1 \vee P_2 \vee P_3$.

(31) For all partitions P_1, P_2 of Y holds $P_1 \wedge P_1 \vee P_2 = P_1$.

(32) For all partitions P_1, P_2 of Y holds $P_1 \vee P_1 \wedge P_2 = P_1$.

(33) For all partitions P_1, P_2, P_3 of Y such that $P_1 \Subset P_3$ and $P_2 \Subset P_3$ holds $P_1 \vee P_2 \Subset P_3$.

(34) For all partitions P_1, P_2, P_3 of Y such that $P_1 \Supset P_3$ and $P_2 \Supset P_3$ holds $P_1 \wedge P_2 \Supset P_3$.

Let us consider Y. The functor $\mathcal{I}(Y)$ yielding a partition of Y is defined as follows:

(Def. 8) $\mathcal{I}(Y) = \mathrm{SmallestPartition}(Y)$.

Let us consider Y. The functor $\mathcal{O}(Y)$ yielding a partition of Y is defined by:

(Def. 9) $\mathcal{O}(Y) = \{Y\}$.

The following propositions are true:

(35) $\mathcal{I}(Y) = \{B : \bigvee_{x\,:\,\mathrm{set}} (B = \{x\} \ \wedge \ x \in Y)\}$.

(36) For every partition P_1 of Y holds $\mathcal{O}(Y) \Supset P_1$ and $P_1 \Supset \mathcal{I}(Y)$.

(37) $\equiv_{\mathcal{O}(Y)} = \nabla_Y$.

(38) $\equiv_{\mathcal{I}(Y)} = \triangle_Y$.

(39) $\mathcal{I}(Y) \Subset \mathcal{O}(Y)$.

(40) For every partition P_1 of Y holds $\mathcal{O}(Y) \vee P_1 = \mathcal{O}(Y)$ and $\mathcal{O}(Y) \wedge P_1 = P_1$.

(41) For every partition P_1 of Y holds $\mathcal{I}(Y) \vee P_1 = P_1$ and $\mathcal{I}(Y) \wedge P_1 = \mathcal{I}(Y)$.

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