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The Definition and Basic Properties of Topological Groups

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The notation and terminology used in this paper are introduced in the following articles: [11], [5], [9], [2], [3], [8], [13], [14], [10], [16], [15], [17], [6], [18], [1], [7], [12], and [4].

1. Preliminaries

For simplicity, we follow the rules: S denotes a 1-sorted structure, R denotes a non empty 1-sorted structure, X denotes a subset of the carrier of R, T denotes a non empty topological structure, and x denotes a set.

Let X, Y be sets. One can verify that every function from X into Y which is bijective is also one-to-one and onto and every function from X into Y which is one-to-one and onto is also bijective.

Let X be a set. Observe that there exists a function from X into X which is one-to-one and onto.

Next we state the proposition

(1) $\mathrm{rng}(\mathrm{id}_S) = \Omega_S$.

Let R be a non empty 1-sorted structure. Note that $(\mathrm{id}_R)^{-1}$ is one-to-one.

We now state two propositions:

(2) $(\mathrm{id}_R)^{-1} = \mathrm{id}_R$.

(3) $(\mathrm{id}_R)^{-1}(X) = X$.

Let S be a 1-sorted structure. One can check that there exists a map from S into S which is one-to-one and onto.

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2. On the Groups

We use the following convention: H denotes a non empty groupoid, P, Q, P_1, Q_1 denote subsets of the carrier of H, and h denotes an element of the carrier of H.

The following propositions are true:

(4) If $P \subseteq P_1$ and $Q \subseteq Q_1$, then $P \cdot Q \subseteq P_1 \cdot Q_1$.

(5) If $P \subseteq Q$, then $P \cdot h \subseteq Q \cdot h$.

(6) If $P \subseteq Q$, then $h \cdot P \subseteq h \cdot Q$.

In the sequel G denotes a group, A, B denote subsets of the carrier of G, and a denotes an element of the carrier of G.

One can prove the following propositions:

(7) $a \in A^{-1}$ iff $a^{-1} \in A$.

(8) $(A^{-1})^{-1} = A$.

(9) $A \subseteq B$ iff $A^{-1} \subseteq B^{-1}$.

(10) $\cdot_G^{-1\circ} A = A^{-1}$.

(11) $\cdot_G^{-1-1}(A) = A^{-1}$.

(12) $\cdot_G^{-1}$ is one-to-one.

(13) $\operatorname{rng} \cdot_G^{-1} =$ the carrier of G.

Let G be a group. Observe that $\cdot_G^{-1}$ is one-to-one and onto.

Next we state two propositions:

(14) $\cdot_G^{-1-1} = \cdot_G^{-1}$.

(15) (The multiplication of $H)^\circ[\!: P, Q :\!] = P \cdot Q$.

Let G be a non empty groupoid and let a be an element of the carrier of G. The functor $a \cdot \Box$ yielding a map from G into G is defined by:

(Def. 1) For every element x of the carrier of G holds $(a \cdot \Box)(x) = a \cdot x$.

The functor $\Box \cdot a$ yields a map from G into G and is defined as follows:

(Def. 2) For every element x of the carrier of G holds $(\Box \cdot a)(x) = x \cdot a$.

Let G be a group and let a be an element of the carrier of G. One can verify that $a \cdot \Box$ is one-to-one and onto and $\Box \cdot a$ is one-to-one and onto.

Next we state four propositions:

(16) $(h \cdot \Box)^\circ P = h \cdot P$.

(17) $(\Box \cdot h)^\circ P = P \cdot h$.

(18) $(a \cdot \Box)^{-1} = a^{-1} \cdot \Box$.

(19) $(\Box \cdot a)^{-1} = \Box \cdot a^{-1}$.

3. On the Topological Spaces

Let T be a non empty topological structure. Observe that $(\mathrm{id}_T)^{-1}$ is continuous.

Next we state the proposition

(20) id_T is a homeomorphism.

Let T be a non empty topological space and let p be a point of T. Observe that every neighbourhood of p is non empty.

Next we state the proposition

(21) For every non empty topological space T and for every point p of T holds Ω_T is a neighbourhood of p.

Let T be a non empty topological space and let p be a point of T. One can check that there exists a neighbourhood of p which is non empty and open.

One can prove the following propositions:

(22) Let S, T be non empty topological spaces and f be a map from S into T. Suppose f is open. Let p be a point of S and P be a neighbourhood of p. Then there exists an open neighbourhood R of $f(p)$ such that $R \subseteq f^{\circ}P$.

(23) Let S, T be non empty topological spaces and f be a map from S into T. Suppose that for every point p of S and for every open neighbourhood P of p there exists a neighbourhood R of $f(p)$ such that $R \subseteq f^{\circ}P$. Then f is open.

(24) Let S, T be non empty topological structures and f be a map from S into T. Then f is a homeomorphism if and only if the following conditions are satisfied:

(i) $\mathrm{dom}\, f = \Omega_S$,

(ii) $\mathrm{rng}\, f = \Omega_T$,

(iii) f is one-to-one, and

(iv) for every subset P of T holds P is closed iff $f^{-1}(P)$ is closed.

(25) Let S, T be non empty topological structures and f be a map from S into T. Then f is a homeomorphism if and only if the following conditions are satisfied:

(i) $\mathrm{dom}\, f = \Omega_S$,

(ii) $\mathrm{rng}\, f = \Omega_T$,

(iii) f is one-to-one, and

(iv) for every subset P of S holds P is open iff $f^{\circ}P$ is open.

(26) Let S, T be non empty topological structures and f be a map from S into T. Then f is a homeomorphism if and only if the following conditions are satisfied:

(i) $\mathrm{dom}\, f = \Omega_S$,

(ii) $\operatorname{rng} f = \Omega_T$,

(iii) f is one-to-one, and

(iv) for every subset P of T holds P is open iff $f^{-1}(P)$ is open.

(27) Let S be a topological space, T be a non empty topological space, and f be a map from S into T. Then f is continuous if and only if for every subset P of the carrier of T holds $f^{-1}(\operatorname{Int} P) \subseteq \operatorname{Int}(f^{-1}(P))$.

Let T be a non empty topological space. One can verify that there exists a subset of T which is non empty and dense.

The following two propositions are true:

(28) Let S, T be non empty topological spaces, f be a map from S into T, and A be a dense subset of S. If f is a homeomorphism, then $f^{\circ}A$ is dense.

(29) Let S, T be non empty topological spaces, f be a map from S into T, and A be a dense subset of T. If f is a homeomorphism, then $f^{-1}(A)$ is dense.

Let S, T be non empty topological structures. Observe that every map from S into T which is homeomorphism is also onto, one-to-one, continuous, and open.

Let T be a non empty topological structure. Observe that there exists a map from T into T which is homeomorphism.

Let T be a non empty topological structure and let f be homeomorphism map from T into T. Note that f^{-1} is homeomorphism.

4. The Group of Homoemorphisms

Let T be a non empty topological structure. A map from T into T is said to be a homeomorphism of T if:

(Def. 3) It is a homeomorphism.

Let T be a non empty topological structure. Then id_T is a homeomorphism of T.

Let T be a non empty topological structure. One can check that every homeomorphism of T is homeomorphism.

We now state two propositions:

(30) For every homeomorphism f of T holds f^{-1} is a homeomorphism of T.

(31) For all homeomorphisms f, g of T holds $f \cdot g$ is a homeomorphism of T.

Let T be a non empty topological structure. The group of homeomorphisms of T is a strict groupoid and is defined by the conditions (Def. 4).

(Def. 4)(i) $x \in$ the carrier of the group of homeomorphisms of T iff x is a homeomorphism of T, and

(ii) for all homeomorphisms f, g of T holds (the multiplication of the group of homeomorphisms of T)$(f, g) = g \cdot f$.

Let T be a non empty topological structure. Note that the group of homeomorphisms of T is non empty.

We now state the proposition

(32) Let f, g be homeomorphisms of T and a, b be elements of the group of homeomorphisms of T. If $f = a$ and $g = b$, then $a \cdot b = g \cdot f$.

Let T be a non empty topological structure. Note that the group of homeomorphisms of T is group-like and associative.

The following two propositions are true:

(33) $\mathrm{id}_T = 1_{\text{the group of homeomorphisms of } T}$.

(34) Let f be a homeomorphism of T and a be an element of the group of homeomorphisms of T. If $f = a$, then $a^{-1} = f^{-1}$.

Let T be a non empty topological structure. We say that T is homogeneous if and only if:

(Def. 5) For all points p, q of T there exists a homeomorphism f of T such that $f(p) = q$.

Let us note that every non empty topological structure which is trivial is also homogeneous.

Let us note that there exists a topological space which is strict, trivial, and non empty.

One can prove the following two propositions:

(35) Let T be a homogeneous non empty topological space. If there exists a point p of T such that $\{p\}$ is closed, then T is a T_1 space.

(36) Let T be a homogeneous non empty topological space. Given a point p of T such that let A be a subset of T. Suppose A is open and $p \in A$. Then there exists a subset B of T such that $p \in B$ and B is open and $\overline{B} \subseteq A$. Then T is a T_3 space.

5. On the Topological Groups

We consider topological group structures as extensions of groupoid and topological structure as systems

⟨ a carrier, a multiplication, a topology ⟩,

where the carrier is a set, the multiplication is a binary operation on the carrier, and the topology is a family of subsets of the carrier.

Let A be a non empty set, let R be a binary operation on A, and let T be a family of subsets of A. Note that $\langle A, R, T \rangle$ is non empty.

Let x be a set, let R be a binary operation on $\{x\}$, and let T be a family of subsets of $\{x\}$. One can verify that $\langle \{x\}, R, T \rangle$ is trivial.

Let us observe that every non empty groupoid which is trivial is also group-like, associative, and commutative.

Let a be a set. Observe that $\{a\}_{\rm top}$ is trivial.

Let us note that there exists a topological group structure which is strict and non empty.

One can verify that there exists a non empty topological group structure which is strict, topological space-like, and trivial.

Let G be a group-like associative non empty topological group structure. Then $\cdot_G^{-1}$ is a map from G into G.

Let G be a group-like associative non empty topological group structure. We say that G is inverse-continuous if and only if:

(Def. 6) $\cdot_G^{-1}$ is continuous.

Let G be a topological space-like topological group structure. We say that G is continuous if and only if:

(Def. 7) For every map f from $[\!: G, G :\!]$ into G such that $f =$ the multiplication of G holds f is continuous.

One can verify that there exists a topological space-like group-like associative non empty topological group structure which is strict, commutative, trivial, inverse-continuous, and continuous.

A semi topological group is a topological space-like group-like associative non empty topological group structure.

A topological group is an inverse-continuous continuous semi topological group.

Next we state several propositions:

(37) Let T be a continuous non empty topological space-like topological group structure, a, b be elements of the carrier of T, and W be a neighbourhood of $a \cdot b$. Then there exists an open neighbourhood A of a and there exists an open neighbourhood B of b such that $A \cdot B \subseteq W$.

(38) Let T be a topological space-like non empty topological group structure. Suppose that for all elements a, b of the carrier of T and for every neighbourhood W of $a \cdot b$ there exists a neighbourhood A of a and there exists a neighbourhood B of b such that $A \cdot B \subseteq W$. Then T is continuous.

(39) Let T be an inverse-continuous semi topological group, a be an element of the carrier of T, and W be a neighbourhood of a^{-1}. Then there exists an open neighbourhood A of a such that $A^{-1} \subseteq W$.

(40) Let T be a semi topological group. Suppose that for every element a of the carrier of T and for every neighbourhood W of a^{-1} there exists a neighbourhood A of a such that $A^{-1} \subseteq W$. Then T is inverse-continuous.

(41) Let T be a topological group, a, b be elements of the carrier of T, and W be a neighbourhood of $a \cdot b^{-1}$. Then there exists an open neighbourhood A of a and there exists an open neighbourhood B of b such that $A \cdot B^{-1} \subseteq W$.

(42) Let T be a semi topological group. Suppose that for all elements a, b of the carrier of T and for every neighbourhood W of $a \cdot b^{-1}$ there exists a neighbourhood A of a and there exists a neighbourhood B of b such that $A \cdot B^{-1} \subseteq W$. Then T is a topological group.

Let G be a continuous non empty topological space-like topological group structure and let a be an element of the carrier of G. One can check that $a \cdot \square$ is continuous and $\square \cdot a$ is continuous.

Next we state two propositions:

(43) Let G be a continuous semi topological group and a be an element of the carrier of G. Then $a \cdot \square$ is a homeomorphism of G.

(44) Let G be a continuous semi topological group and a be an element of the carrier of G. Then $\square \cdot a$ is a homeomorphism of G.

The following proposition is true

(45) For every inverse-continuous semi topological group G holds $\cdot_{G}^{-1}$ is a homeomorphism of G.

One can verify that every semi topological group which is continuous is also homogeneous.

The following two propositions are true:

(46) Let G be a continuous semi topological group, F be a closed subset of G, and a be an element of the carrier of G. Then $F \cdot a$ is closed.

(47) Let G be a continuous semi topological group, F be a closed subset of G, and a be an element of the carrier of G. Then $a \cdot F$ is closed.

We now state the proposition

(48) For every inverse-continuous semi topological group G and for every closed subset F of G holds F^{-1} is closed.

The following two propositions are true:

(49) Let G be a continuous semi topological group, O be an open subset of G, and a be an element of the carrier of G. Then $O \cdot a$ is open.

(50) Let G be a continuous semi topological group, O be an open subset of G, and a be an element of the carrier of G. Then $a \cdot O$ is open.

We now state the proposition

(51) For every inverse-continuous semi topological group G and for every open subset O of G holds O^{-1} is open.

The following two propositions are true:

(52) For every continuous semi topological group G and for all subsets A, O of G such that O is open holds $O \cdot A$ is open.

(53) For every continuous semi topological group G and for all subsets A, O of G such that O is open holds $A \cdot O$ is open.

One can prove the following propositions:

(54) Let G be an inverse-continuous semi topological group, a be a point of G, and A be a neighbourhood of a. Then A^{-1} is a neighbourhood of a^{-1}.

(55) Let G be a topological group, a be a point of G, and A be a neighbourhood of $a \cdot a^{-1}$. Then there exists an open neighbourhood B of a such that $B \cdot B^{-1} \subseteq A$.

(56) For every inverse-continuous semi topological group G and for every dense subset A of G holds A^{-1} is dense.

We now state two propositions:

(57) Let G be a continuous semi topological group, A be a dense subset of G, and a be a point of G. Then $a \cdot A$ is dense.

(58) Let G be a continuous semi topological group, A be a dense subset of G, and a be a point of G. Then $A \cdot a$ is dense.

We now state two propositions:

(59) Let G be a topological group, B be a basis of 1_G, and M be a dense subset of G. Then $\{V \cdot x; V$ ranges over subsets of the carrier of G, x ranges over points of G: $V \in B \ \wedge \ x \in M\}$ is a basis of G.

(60) Every topological group is a T_3 space.

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