FORMALIZED MATHEMATICS
Volume 7, Number 2, 1998
University of Białystok

Lawson Topology in Continuous Lattices[1]

Grzegorz Bancerek
University of Białystok

Summary. The article completes Mizar formalization of Section 1 of Chapter III of [9, pp. 145–147].

MML Identifier: `WAYBEL21`.

The articles [8], [7], [1], [16], [10], [13], [17], [15], [11], [6], [3], [4], [12], [2], [18], [14], and [5] provide the terminology and notation for this paper.

1. Semilattice Homomorphism and Inheritance

Let S, T be semilattices. Let us assume that if S is upper-bounded, then T is upper-bounded. A map from S into T is said to be a semilattice morphism from S into T if:

(Def. 1) For every finite subset X of S holds it preserves inf of X.

Let S, T be semilattices. One can check that every map from S into T which is meet-preserving is also monotone.

Let S be a semilattice and let T be an upper-bounded semilattice. One can check that every semilattice morphism from S into T is meet-preserving.

Next we state a number of propositions:

(1) For all upper-bounded semilattices S, T and for every semilattice morphism f from S into T holds $f(\top_S) = \top_T$.

(2) Let S, T be semilattices and f be a map from S into T. Suppose f is meet-preserving. Let X be a finite non empty subset of S. Then f preserves inf of X.

[1]Partially supported by NATO Grant CRG 951368, NSERC OGP 9207 grant and KBN grant 8 T11C 018 12.

ISSN 1426–2630

(3) Let S, T be upper-bounded semilattices and f be a meet-preserving map from S into T. If $f(\top_S) = \top_T$, then f is a semilattice morphism from S into T.

(4) Let S, T be semilattices and f be a map from S into T. Suppose f is meet-preserving and for every filtered non empty subset X of S holds f preserves inf of X. Let X be a non empty subset of S. Then f preserves inf of X.

(5) Let S, T be semilattices and f be a map from S into T. Suppose f is infs-preserving. Then f is a semilattice morphism from S into T.

(6) Let S_1, T_1, S_2, T_2 be non empty relational structures. Suppose that
(i) the relational structure of S_1 = the relational structure of S_2, and
(ii) the relational structure of T_1 = the relational structure of T_2.
Let f_1 be a map from S_1 into T_1 and f_2 be a map from S_2 into T_2 such that $f_1 = f_2$. Then
(iii) if f_1 is infs-preserving, then f_2 is infs-preserving, and
(iv) if f_1 is directed-sups-preserving, then f_2 is directed-sups-preserving.

(7) Let S_1, T_1, S_2, T_2 be non empty relational structures. Suppose that
(i) the relational structure of S_1 = the relational structure of S_2, and
(ii) the relational structure of T_1 = the relational structure of T_2.
Let f_1 be a map from S_1 into T_1 and f_2 be a map from S_2 into T_2 such that $f_1 = f_2$. Then
(iii) if f_1 is sups-preserving, then f_2 is sups-preserving, and
(iv) if f_1 is filtered-infs-preserving, then f_2 is filtered-infs-preserving.

(8) Let T be a complete lattice and S be an infs-inheriting full non empty relational substructure of T. Then $\mathrm{incl}(S,T)$ is infs-preserving.

(9) Let T be a complete lattice and S be a sups-inheriting full non empty relational substructure of T. Then $\mathrm{incl}(S,T)$ is sups-preserving.

(10) Let T be an up-complete non empty poset and S be a directed-sups-inheriting full non empty relational substructure of T. Then $\mathrm{incl}(S,T)$ is directed-sups-preserving.

(11) Let T be a complete lattice and S be a filtered-infs-inheriting full non empty relational substructure of T. Then $\mathrm{incl}(S,T)$ is filtered-infs-preserving.

(12) Let T_1, T_2, R be relational structures and S be a relational substructure of T_1. Suppose that
(i) the relational structure of T_1 = the relational structure of T_2, and
(ii) the relational structure of S = the relational structure of R.
Then R is a relational substructure of T_2 and if S is full, then R is a full relational substructure of T_2.

(13) Every non empty relational structure T is an infs-inheriting sups-inheriting full relational substructure of T.

Let T be a complete lattice. Observe that there exists a continuous subframe of T which is complete.

We now state a number of propositions:

(14) Let T be a semilattice and S be a full non empty relational substructure of T. Then S is meet-inheriting if and only if for every finite non empty subset X of S holds $\bigsqcap_T X \in$ the carrier of S.

(15) Let T be a sup-semilattice and S be a full non empty relational substructure of T. Then S is join-inheriting if and only if for every finite non empty subset X of S holds $\bigsqcup_T X \in$ the carrier of S.

(16) Let T be an upper-bounded semilattice and S be a meet-inheriting full non empty relational substructure of T. Suppose $\top_T \in$ the carrier of S and S is filtered-infs-inheriting. Then S is infs-inheriting.

(17) Let T be a lower-bounded sup-semilattice and S be a join-inheriting full non empty relational substructure of T. Suppose $\bot_T \in$ the carrier of S and S is directed-sups-inheriting. Then S is sups-inheriting.

(18) Let T be a complete lattice and S be a full non empty relational substructure of T. If S is infs-inheriting, then S is complete.

(19) Let T be a complete lattice and S be a full non empty relational substructure of T. If S is sups-inheriting, then S is complete.

(20) Let T_1, T_2 be non empty relational structures, S_1 be a non empty full relational substructure of T_1, and S_2 be a non empty full relational substructure of T_2. Suppose that
   (i) the relational structure of $T_1 =$ the relational structure of T_2, and
   (ii) the carrier of $S_1 =$ the carrier of S_2.

   If S_1 is infs-inheriting, then S_2 is infs-inheriting.

(21) Let T_1, T_2 be non empty relational structures, S_1 be a non empty full relational substructure of T_1, and S_2 be a non empty full relational substructure of T_2. Suppose that
   (i) the relational structure of $T_1 =$ the relational structure of T_2, and
   (ii) the carrier of $S_1 =$ the carrier of S_2.

   If S_1 is sups-inheriting, then S_2 is sups-inheriting.

(22) Let T_1, T_2 be non empty relational structures, S_1 be a non empty full relational substructure of T_1, and S_2 be a non empty full relational substructure of T_2. Suppose that
   (i) the relational structure of $T_1 =$ the relational structure of T_2, and
   (ii) the carrier of $S_1 =$ the carrier of S_2.

   If S_1 is directed-sups-inheriting, then S_2 is directed-sups-inheriting.

(23) Let T_1, T_2 be non empty relational structures, S_1 be a non empty full relational substructure of T_1, and S_2 be a non empty full relational substructure of T_2. Suppose that

(i) the relational structure of T_1 = the relational structure of T_2, and

(ii) the carrier of S_1 = the carrier of S_2.

If S_1 is filtered-infs-inheriting, then S_2 is filtered-infs-inheriting.

2. Nets and Limits

The following proposition is true

(24) Let S, T be non empty topological spaces, N be a net in S, and f be a map from S into T. If f is continuous, then $f^\circ \operatorname{Lim} N \subseteq \operatorname{Lim}(f \cdot N)$.

Let T be a non empty relational structure and let N be a non empty net structure over T. Let us observe that N is antitone if and only if:

(Def. 2) For all elements i, j of N such that $i \leqslant j$ holds $N(i) \geqslant N(j)$.

Let T be a non empty reflexive relational structure and let x be an element of T. Observe that $\langle \{x\}^{\mathrm{op}}; \mathrm{id} \rangle$ is transitive directed monotone and antitone.

Let T be a non empty reflexive relational structure. Note that there exists a net in T which is monotone, antitone, reflexive, and strict.

Let T be a non empty relational structure and let F be a non empty subset of T. Note that $\langle F^{\mathrm{op}}; \mathrm{id} \rangle$ is antitone.

Let S, T be non empty reflexive relational structures, let f be a monotone map from S into T, and let N be an antitone non empty net structure over S. Note that $f \cdot N$ is antitone.

We now state a number of propositions:

(25) Let S be a complete lattice and N be a net in S. Then $\{\bigsqcap_S \{N(i); i$ ranges over elements of the carrier of N: $i \geqslant j\}$: j ranges over elements of the carrier of $N\}$ is a directed non empty subset of S.

(26) Let S be a non empty poset and N be a monotone reflexive net in S. Then $\{\bigsqcap_S \{N(i); i$ ranges over elements of the carrier of N: $i \geqslant j\}$: j ranges over elements of the carrier of $N\}$ is a directed non empty subset of S.

(27) Let S be a non empty 1-sorted structure, N be a non empty net structure over S, and X be a set. If rng (the mapping of N) $\subseteq X$, then N is eventually in X.

(28) For every inf-complete non empty poset R and for every non empty filtered subset F of R holds $\liminf \langle F^{\mathrm{op}}; \mathrm{id} \rangle = \inf F$.

(29) Let S, T be inf-complete non empty posets, X be a non empty filtered subset of S, and f be a monotone map from S into T. Then $\liminf(f \cdot \langle X^{\rm op}; {\rm id}\rangle) = \inf(f^\circ X)$.

(30) Let S, T be non empty top-posets, X be a non empty filtered subset of S, f be a monotone map from S into T, and Y be a non empty filtered subset of T. If $Y = f^\circ X$, then $f \cdot \langle X^{\rm op}; {\rm id}\rangle$ is a subnet of $\langle Y^{\rm op}; {\rm id}\rangle$.

(31) Let S, T be non empty top-posets, X be a non empty filtered subset of S, f be a monotone map from S into T, and Y be a non empty filtered subset of T. If $Y = f^\circ X$, then ${\rm Lim}\langle Y^{\rm op}; {\rm id}\rangle \subseteq {\rm Lim}(f \cdot \langle X^{\rm op}; {\rm id}\rangle)$.

(32) Let S be a non empty reflexive relational structure and D be a non empty subset of S. Then the mapping of ${\rm NetStr}(D) = {\rm id}_D$ and the carrier of ${\rm NetStr}(D) = D$ and ${\rm NetStr}(D)$ is a full relational substructure of S.

(33) Let S, T be up-complete non empty posets, f be a monotone map from S into T, and D be a non empty directed subset of S. Then $\liminf(f \cdot {\rm NetStr}(D)) = \sup(f^\circ D)$.

(34) Let S be a non empty reflexive relational structure, D be a non empty directed subset of S, and i, j be elements of ${\rm NetStr}(D)$. Then $i \leqslant j$ if and only if $({\rm NetStr}(D))(i) \leqslant ({\rm NetStr}(D))(j)$.

(35) For every Lawson complete top-lattice T and for every directed non empty subset D of T holds $\sup D \in {\rm Lim}\,{\rm NetStr}(D)$.

Let T be a non empty 1-sorted structure, let N be a net in T, and let M be a non empty net structure over T. Let us assume that M is a subnet of N. A map from M into N is said to be a embedding of M into N if it satisfies the conditions (Def. 3).

(Def. 3)(i) The mapping of $M =$ (the mapping of N) $\cdot$ it, and

(ii) for every element m of N there exists an element n of M such that for every element p of M such that $n \leqslant p$ holds $m \leqslant {\rm it}(p)$.

One can prove the following propositions:

(36) Let T be a non empty 1-sorted structure, N be a net in T, M be a non empty subnet of N, e be a embedding of M into N, and i be an element of M. Then $M(i) = N(e(i))$.

(37) For every complete lattice T and for every net N in T and for every subnet M of N holds $\liminf N \leqslant \liminf M$.

(38) Let T be a complete lattice, N be a net in T, M be a subnet of N, and e be a embedding of M into N. Suppose that for every element i of N and for every element j of M such that $e(j) \leqslant i$ there exists an element j' of M such that $j' \geqslant j$ and $N(i) \geqslant M(j')$. Then $\liminf N = \liminf M$.

(39) Let T be a non empty relational structure, N be a net in T, and M be a non empty full structure of a subnet of N. Suppose that for every element i of N there exists an element j of N such that $j \geqslant i$ and $j \in$ the carrier

of M. Then M is a subnet of N and $\mathrm{incl}(M, N)$ is a embedding of M into N.

(40) Let T be a non empty relational structure, N be a net in T, and i be an element of N. Then $N{\restriction}i$ is a subnet of N and $\mathrm{incl}(N{\restriction}i, N)$ is a embedding of $N{\restriction}i$ into N.

(41) For every complete lattice T and for every net N in T and for every element i of N holds $\liminf(N{\restriction}i) = \liminf N$.

(42) Let T be a non empty relational structure, N be a net in T, and X be a set. Suppose N is eventually in X. Then there exists an element i of N such that $N(i) \in X$ and rng (the mapping of $N{\restriction}i) \subseteq X$.

(43) Let T be a Lawson complete top-lattice and N be an eventually-filtered net in T. Then rng (the mapping of N) is a filtered non empty subset of T.

(44) For every Lawson complete top-lattice T and for every eventually-filtered net N in T holds $\mathrm{Lim}\, N = \{\inf N\}$.

3. Lawson Topology Revisited

One can prove the following propositions:

(45) Let S, T be Lawson complete top-lattices and f be a meet-preserving map from S into T. Then f is continuous if and only if the following conditions are satisfied:

(i) f is directed-sups-preserving, and

(ii) for every non empty subset X of S holds f preserves inf of X.

(46) Let S, T be Lawson complete top-lattices and f be a semilattice morphism from S into T. Then f is continuous if and only if f is infs-preserving and directed-sups-preserving.

Let S, T be non empty relational structures and let f be a map from S into T. We say that f is liminfs-preserving if and only if:

(Def. 4) For every net N in S holds $f(\liminf N) = \liminf(f \cdot N)$.

One can prove the following propositions:

(47) Let S, T be Lawson complete top-lattices and f be a semilattice morphism from S into T. Then f is continuous if and only if f is liminfs-preserving.

(48) Let T be a Lawson complete continuous top-lattice and S be a meet-inheriting full non empty relational substructure of T. Suppose $\top_T \in$ the carrier of S and there exists a subset X of T such that $X =$ the carrier of S and X is closed. Then S is infs-inheriting.

(49) Let T be a Lawson complete continuous top-lattice and S be a full non empty relational substructure of T. Given a subset X of T such that $X =$ the carrier of S and X is closed. Then S is directed-sups-inheriting.

(50) Let T be a Lawson complete continuous top-lattice and S be an infs-inheriting directed-sups-inheriting full non empty relational substructure of T. Then there exists a subset X of T such that $X =$ the carrier of S and X is closed.

(51) Let T be a Lawson complete continuous top-lattice, S be an infs-inheriting directed-sups-inheriting full non empty relational substructure of T, and N be a net in T. If N is eventually in the carrier of S, then $\liminf N \in$ the carrier of S.

(52) Let T be a Lawson complete continuous top-lattice and S be a meet-inheriting full non empty relational substructure of T. Suppose that
(i) $\top_T \in$ the carrier of S, and
(ii) for every net N in T such that rng (the mapping of N) $\subseteq$ the carrier of S holds $\liminf N \in$ the carrier of S.
Then S is infs-inheriting.

(53) Let T be a Lawson complete continuous top-lattice and S be a full non empty relational substructure of T. Suppose that for every net N in T such that rng (the mapping of N) $\subseteq$ the carrier of S holds $\liminf N \in$ the carrier of S. Then S is directed-sups-inheriting.

(54) Let T be a Lawson complete continuous top-lattice, S be a meet-inheriting full non empty relational substructure of T, and X be a subset of T. Suppose $X =$ the carrier of S and $\top_T \in X$. Then X is closed if and only if for every net N in T such that N is eventually in X holds $\liminf N \in X$.

References

[1] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(**5**):719–725, 1991.
[2] Grzegorz Bancerek. Bounds in posets and relational substructures. *Formalized Mathematics*, 6(**1**):81–91, 1997.
[3] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(**1**):93–107, 1997.
[4] Grzegorz Bancerek. The "way-below" relation. *Formalized Mathematics*, 6(**1**):169–176, 1997.
[5] Grzegorz Bancerek. Bases and refinements of topologies. *Formalized Mathematics*, 7(**1**):35–43, 1998.
[6] Grzegorz Bancerek. The Lawson topology. *Formalized Mathematics*, 7(**2**):163–168, 1998.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[9] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, Berlin, Heidelberg, New York, 1980.
[10] Adam Grabowski. On the category of posets. *Formalized Mathematics*, 5(**4**):501–505, 1996.
[11] Adam Grabowski. Scott-continuous functions. *Formalized Mathematics*, 7(**1**):13–18, 1998.

[12] Artur Korniłowicz. On the topological properties of meet-continuous lattices. *Formalized Mathematics*, 6(**2**):269–277, 1997.
[13] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[14] Andrzej Trybulec. Moore-Smith convergence. *Formalized Mathematics*, 6(**2**):213–225, 1997.
[15] Andrzej Trybulec. Scott topology. *Formalized Mathematics*, 6(**2**):311–319, 1997.
[16] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[18] Mariusz Żynel and Czesław Byliński. Properties of relational structures, posets, lattices and maps. *Formalized Mathematics*, 6(**1**):123–130, 1997.

Received July 12, 1998