FORMALIZED MATHEMATICS
Volume 7, Number 2, 1998
University of Białystok

Bases of Continuous Lattices[1]

Robert Milewski
University of Białystok

Summary. The article is a Mizar formalization of [7, 168–169]. We show definition and fundamental theorems from theory of basis of continuous lattices.

MML Identifier: `WAYBEL23`.

The terminology and notation used in this paper are introduced in the following articles: [13], [5], [1], [11], [8], [14], [12], [3], [6], [4], [10], [2], [9], and [15].

1. Preliminaries

The following proposition is true

(1) For every non empty poset L and for every element x of L holds compactbelow$(x) = {\downarrow}x \cap$ the carrier of CompactSublatt(L).

Let L be a non empty reflexive transitive relational structure and let X be a subset of $\langle \mathrm{Ids}(L), \subseteq \rangle$. Then $\bigcup X$ is a subset of L.

The following propositions are true:

(2) For every non empty relational structure L and for all subsets X, Y of the carrier of L such that $X \subseteq Y$ holds finsups$(X) \subseteq$ finsups(Y).

(3) Let L be a non empty transitive relational structure, S be a sups-inheriting non empty full relational substructure of L, X be a subset of the carrier of L, and Y be a subset of the carrier of S. If $X = Y$, then finsups$(X) \subseteq$ finsups(Y).

(4) Let L be a complete transitive antisymmetric non empty relational structure, S be a sups-inheriting non empty full relational substructure of L, X be a subset of the carrier of L, and Y be a subset of the carrier of S. If $X = Y$, then finsups$(X) =$ finsups(Y).

[1]This work has been supported by KBN Grant 8 T11C 018 12.

ISSN 1426–2630

(5) Let L be a complete sup-semilattice and S be a join-inheriting non empty full relational substructure of L. Suppose $\bot_L \in$ the carrier of S. Let X be a subset of L and Y be a subset of S. If $X = Y$, then $\mathrm{finsups}(Y) \subseteq \mathrm{finsups}(X)$.

(6) For every lower-bounded sup-semilattice L and for every subset X of $\langle \mathrm{Ids}(L), \subseteq \rangle$ holds $\sup X = {\downarrow}\mathrm{finsups}(\bigcup X)$.

(7) For every reflexive transitive relational structure L and for every subset X of L holds ${\downarrow}{\downarrow}X = {\downarrow}X$.

(8) For every reflexive transitive relational structure L and for every subset X of L holds ${\uparrow}{\uparrow}X = {\uparrow}X$.

(9) For every non empty reflexive transitive relational structure L and for every element x of L holds ${\downarrow}{\downarrow}x = {\downarrow}x$.

(10) For every non empty reflexive transitive relational structure L and for every element x of L holds ${\uparrow}{\uparrow}x = {\uparrow}x$.

(11) Let L be a non empty relational structure, S be a non empty relational substructure of L, X be a subset of L, and Y be a subset of S. If $X = Y$, then ${\downarrow}Y \subseteq {\downarrow}X$.

(12) Let L be a non empty relational structure, S be a non empty relational substructure of L, X be a subset of L, and Y be a subset of S. If $X = Y$, then ${\uparrow}Y \subseteq {\uparrow}X$.

(13) Let L be a non empty relational structure, S be a non empty relational substructure of L, x be an element of L, and y be an element of S. If $x = y$, then ${\downarrow}y \subseteq {\downarrow}x$.

(14) Let L be a non empty relational structure, S be a non empty relational substructure of L, x be an element of L, and y be an element of S. If $x = y$, then ${\uparrow}y \subseteq {\uparrow}x$.

2. Relational Subsets

Let L be a non empty relational structure and let S be a subset of L. We say that S is meet-closed if and only if:

(Def. 1) $\mathrm{sub}(S)$ is meet-inheriting.

Let L be a non empty relational structure and let S be a subset of L. We say that S is join-closed if and only if:

(Def. 2) $\mathrm{sub}(S)$ is join-inheriting.

Let L be a non empty relational structure and let S be a subset of L. We say that S is infs-closed if and only if:

(Def. 3) $\mathrm{sub}(S)$ is infs-inheriting.

Let L be a non empty relational structure and let S be a subset of L. We say that S is sups-closed if and only if:

(Def. 4) sub(S) is sups-inheriting.

Let L be a non empty relational structure. Observe that every subset of L which is infs-closed is also meet-closed and every subset of L which is sups-closed is also join-closed.

Let L be a non empty relational structure. One can verify that there exists a subset of L which is infs-closed, sups-closed, and non empty.

One can prove the following propositions:

(15) Let L be a non empty relational structure and S be a subset of L. Then S is meet-closed if and only if for all elements x, y of L such that $x \in S$ and $y \in S$ and inf $\{x, y\}$ exists in L holds $\inf\{x, y\} \in S$.

(16) Let L be a non empty relational structure and S be a subset of L. Then S is join-closed if and only if for all elements x, y of L such that $x \in S$ and $y \in S$ and sup $\{x, y\}$ exists in L holds $\sup\{x, y\} \in S$.

(17) Let L be an antisymmetric relational structure with g.l.b.'s and S be a subset of L. Then S is meet-closed if and only if for all elements x, y of L such that $x \in S$ and $y \in S$ holds $\inf\{x, y\} \in S$.

(18) Let L be an antisymmetric relational structure with l.u.b.'s and S be a subset of L. Then S is join-closed if and only if for all elements x, y of L such that $x \in S$ and $y \in S$ holds $\sup\{x, y\} \in S$.

(19) Let L be a non empty relational structure and S be a subset of L. Then S is infs-closed if and only if for every subset X of S such that inf X exists in L holds $\bigsqcap_L X \in S$.

(20) Let L be a non empty relational structure and S be a subset of L. Then S is sups-closed if and only if for every subset X of S such that sup X exists in L holds $\bigsqcup_L X \in S$.

(21) Let L be a non empty transitive relational structure, S be an infs-closed non empty subset of L, and X be a subset of S. If inf X exists in L, then inf X exists in sub(S) and $\bigsqcap_{\mathrm{sub}(S)} X = \bigsqcap_L X$.

(22) Let L be a non empty transitive relational structure, S be a sups-closed non empty subset of L, and X be a subset of S. If sup X exists in L, then sup X exists in sub(S) and $\bigsqcup_{\mathrm{sub}(S)} X = \bigsqcup_L X$.

(23) Let L be a non empty transitive relational structure, S be a meet-closed non empty subset of L, and x, y be elements of S. Suppose inf $\{x, y\}$ exists in L. Then inf $\{x, y\}$ exists in sub(S) and $\bigsqcap_{\mathrm{sub}(S)}\{x, y\} = \bigsqcap_L\{x, y\}$.

(24) Let L be a non empty transitive relational structure, S be a join-closed non empty subset of L, and x, y be elements of S. Suppose sup $\{x, y\}$ exists in L. Then sup $\{x, y\}$ exists in sub(S) and $\bigsqcup_{\mathrm{sub}(S)}\{x, y\} = \bigsqcup_L\{x, y\}$.

(25) Let L be an antisymmetric transitive relational structure with g.l.b.'s and S be a non empty meet-closed subset of L. Then $\mathrm{sub}(S)$ has g.l.b.'s.

(26) Let L be an antisymmetric transitive relational structure with l.u.b.'s and S be a non empty join-closed subset of L. Then $\mathrm{sub}(S)$ has l.u.b.'s.

Let L be an antisymmetric transitive relational structure with g.l.b.'s and let S be a non empty meet-closed subset of L. Observe that $\mathrm{sub}(S)$ has g.l.b.'s.

Let L be an antisymmetric transitive relational structure with l.u.b.'s and let S be a non empty join-closed subset of L. Observe that $\mathrm{sub}(S)$ has l.u.b.'s.

The following four propositions are true:

(27) Let L be a complete transitive antisymmetric non empty relational structure, S be an infs-closed non empty subset of L, and X be a subset of S. Then $\bigsqcap_{\mathrm{sub}(S)} X = \bigsqcap_L X$.

(28) Let L be a complete transitive antisymmetric non empty relational structure, S be a sups-closed non empty subset of L, and X be a subset of S. Then $\bigsqcup_{\mathrm{sub}(S)} X = \bigsqcup_L X$.

(29) For every semilattice L holds every meet-closed subset of L is filtered.

(30) For every sup-semilattice L holds every join-closed subset of L is directed.

Let L be a semilattice. Observe that every subset of L which is meet-closed is also filtered.

Let L be a sup-semilattice. One can check that every subset of L which is join-closed is also directed.

The following propositions are true:

(31) Let L be a semilattice and S be an upper non empty subset of L. Then S is a filter of L if and only if S is meet-closed.

(32) Let L be a sup-semilattice and S be a lower non empty subset of L. Then S is an ideal of L if and only if S is join-closed.

(33) For every non empty relational structure L and for all join-closed subsets S_1, S_2 of L holds $S_1 \cap S_2$ is join-closed.

(34) For every non empty relational structure L and for all meet-closed subsets S_1, S_2 of L holds $S_1 \cap S_2$ is meet-closed.

(35) For every sup-semilattice L and for every element x of the carrier of L holds $\downarrow x$ is join-closed.

(36) For every semilattice L and for every element x of the carrier of L holds $\downarrow x$ is meet-closed.

(37) For every sup-semilattice L and for every element x of the carrier of L holds $\uparrow x$ is join-closed.

(38) For every semilattice L and for every element x of the carrier of L holds $\uparrow x$ is meet-closed.

Let L be a sup-semilattice and let x be an element of L. Observe that $\downarrow x$ is join-closed and $\uparrow x$ is join-closed.

Let L be a semilattice and let x be an element of L. Note that $\downarrow x$ is meet-closed and $\uparrow x$ is meet-closed.

Next we state three propositions:

(39) For every sup-semilattice L and for every element x of L holds $\twoheaddownarrow x$ is join-closed.

(40) For every semilattice L and for every element x of L holds $\twoheaddownarrow x$ is meet-closed.

(41) For every sup-semilattice L and for every element x of L holds $\twoheaduparrow x$ is join-closed.

Let L be a sup-semilattice and let x be an element of L. Note that $\twoheaddownarrow x$ is join-closed and $\twoheaduparrow x$ is join-closed.

Let L be a semilattice and let x be an element of L. Observe that $\twoheaddownarrow x$ is meet-closed.

3. About Bases of Continuous Lattices

Let T be a topological structure. The functor weight T yields a cardinal number and is defined as follows:

(Def. 5) $\text{weight}\, T = \bigcap\{\overline{\overline{B}} : B \text{ ranges over bases of } T\}$.

Let T be a topological structure. We say that T is second-countable if and only if:

(Def. 6) $\text{weight}\, T \subseteq \omega$.

Let L be a continuous sup-semilattice. A subset of L is called a CLbasis of L if:

(Def. 7) It is join-closed and for every element x of L holds $x = \sup(\twoheaddownarrow x \cap \text{it})$.

Let L be a non empty relational structure and let S be a subset of L. We say that S has bottom if and only if:

(Def. 8) $\bot_L \in S$.

Let L be a non empty relational structure and let S be a subset of L. We say that S has top if and only if:

(Def. 9) $\top_L \in S$.

Let L be a non empty relational structure. Note that every subset of L which has bottom is non empty.

Let L be a non empty relational structure. Observe that every subset of L which has top is non empty.

Let L be a non empty relational structure. Note that there exists a subset of L which has bottom and there exists a subset of L which has top.

Let L be a continuous sup-semilattice. One can verify that there exists a CLbasis of L which has bottom and there exists a CLbasis of L which has top.

One can prove the following proposition

(42) Let L be a lower-bounded antisymmetric non empty relational structure and S be a subset of L with bottom. Then $\mathrm{sub}(S)$ is lower-bounded.

Let L be a lower-bounded antisymmetric non empty relational structure and let S be a subset of L with bottom. One can verify that $\mathrm{sub}(S)$ is lower-bounded.

Let L be a continuous sup-semilattice. Observe that every CLbasis of L is join-closed.

One can check that there exists a continuous lattice which is bounded and non trivial.

Let L be a lower-bounded non trivial continuous sup-semilattice. One can verify that every CLbasis of L is non empty.

One can prove the following propositions:

(43) For every sup-semilattice L holds the carrier of CompactSublatt(L) is a join-closed subset of L.

(44) For every algebraic lower-bounded lattice L holds the carrier of CompactSublatt(L) is a CLbasis of L with bottom.

(45) Let L be a continuous lower-bounded sup-semilattice. If the carrier of CompactSublatt(L) is a CLbasis of L, then L is algebraic.

(46) Let L be a continuous lower-bounded lattice and B be a join-closed subset of L. Then B is a CLbasis of L if and only if for all elements x, y of L such that $y \nleqslant x$ there exists an element b of L such that $b \in B$ and $b \nleqslant x$ and $b \ll y$.

(47) Let L be a continuous lower-bounded lattice and B be a join-closed subset of L. Suppose $\bot_L \in B$. Then B is a CLbasis of L if and only if for all elements x, y of L such that $x \ll y$ there exists an element b of L such that $b \in B$ and $x \leqslant b$ and $b \ll y$.

(48) Let L be a continuous lower-bounded lattice and B be a join-closed subset of L. Suppose $\bot_L \in B$. Then B is a CLbasis of L if and only if the following conditions are satisfied:

(i) the carrier of CompactSublatt(L) $\subseteq B$, and

(ii) for all elements x, y of L such that $y \nleqslant x$ there exists an element b of L such that $b \in B$ and $b \nleqslant x$ and $b \leqslant y$.

(49) Let L be a continuous lower-bounded lattice and B be a join-closed subset of L. Suppose $\bot_L \in B$. Then B is a CLbasis of L if and only if for all elements x, y of L such that $y \nleqslant x$ there exists an element b of L such that $b \in B$ and $b \nleqslant x$ and $b \leqslant y$.

(50) Let L be a lower-bounded sup-semilattice and S be a non empty full relational substructure of L. Suppose $\bot_L \in$ the carrier of S and the carrier

of S is a join-closed subset of L. Let x be an element of L. Then $\downarrow x \cap$ the carrier of S is an ideal of S.

Let L be a non empty reflexive transitive relational structure and let S be a non empty full relational substructure of L. The functor $\mathrm{supMap}\, S$ yielding a map from $\langle \mathrm{Ids}(S), \subseteq \rangle$ into L is defined by:

(Def. 10) For every ideal I of S holds $(\mathrm{supMap}\, S)(I) = \bigsqcup_L I$.

Let L be a non empty reflexive transitive relational structure and let S be a non empty full relational substructure of L. The functor $\mathrm{idsMap}\, S$ yields a map from $\langle \mathrm{Ids}(S), \subseteq \rangle$ into $\langle \mathrm{Ids}(L), \subseteq \rangle$ and is defined by:

(Def. 11) For every ideal I of S there exists a subset J of L such that $I = J$ and $(\mathrm{idsMap}\, S)(I) = \downarrow J$.

Let L be a non empty relational structure and let B be a non empty subset of the carrier of L. Observe that $\mathrm{sub}(B)$ is non empty.

Let L be a reflexive relational structure and let B be a subset of the carrier of L. Note that $\mathrm{sub}(B)$ is reflexive.

Let L be a transitive relational structure and let B be a subset of the carrier of L. Note that $\mathrm{sub}(B)$ is transitive.

Let L be an antisymmetric relational structure and let B be a subset of the carrier of L. Observe that $\mathrm{sub}(B)$ is antisymmetric.

Let L be a lower-bounded continuous sup-semilattice and let B be a CLbasis of L with bottom. The functor $\mathrm{baseMap}\, B$ yielding a map from L into $\langle \mathrm{Ids}(\mathrm{sub}(B)), \subseteq \rangle$ is defined as follows:

(Def. 12) For every element x of L holds $(\mathrm{baseMap}\, B)(x) = \downarrow x \cap B$.

We now state a number of propositions:

(51) Let L be a non empty reflexive transitive relational structure and S be a non empty full relational substructure of L. Then $\mathrm{dom}\, \mathrm{supMap}\, S = \mathrm{Ids}(S)$ and $\mathrm{rng}\, \mathrm{supMap}\, S$ is a subset of L.

(52) Let L be a non empty reflexive transitive relational structure, S be a non empty full relational substructure of L, and x be a set. Then $x \in \mathrm{dom}\, \mathrm{supMap}\, S$ if and only if x is an ideal of S.

(53) Let L be a non empty reflexive transitive relational structure and S be a non empty full relational substructure of L. Then $\mathrm{dom}\, \mathrm{idsMap}\, S = \mathrm{Ids}(S)$ and $\mathrm{rng}\, \mathrm{idsMap}\, S$ is a subset of $\mathrm{Ids}(L)$.

(54) Let L be a non empty reflexive transitive relational structure, S be a non empty full relational substructure of L, and x be a set. Then $x \in \mathrm{dom}\, \mathrm{idsMap}\, S$ if and only if x is an ideal of S.

(55) Let L be a non empty reflexive transitive relational structure, S be a non empty full relational substructure of L, and x be a set. If $x \in \mathrm{rng}\, \mathrm{idsMap}\, S$, then x is an ideal of L.

(56) Let L be a lower-bounded continuous sup-semilattice and B be a CLbasis of L with bottom. Then dom baseMap B = the carrier of L and rng baseMap B is a subset of Ids(sub(B)).

(57) Let L be a lower-bounded continuous sup-semilattice, B be a CLbasis of L with bottom, and x be a set. If $x \in$ rng baseMap B, then x is an ideal of sub(B).

(58) For every up-complete non empty poset L and for every non empty full relational substructure S of L holds supMap S is monotone.

(59) Let L be a non empty reflexive transitive relational structure and S be a non empty full relational substructure of L. Then idsMap S is monotone.

(60) For every lower-bounded continuous sup-semilattice L and for every CLbasis B of L with bottom holds baseMap B is monotone.

Let L be an up-complete non empty poset and let S be a non empty full relational substructure of L. Observe that supMap S is monotone.

Let L be a non empty reflexive transitive relational structure and let S be a non empty full relational substructure of L. One can check that idsMap S is monotone.

Let L be a lower-bounded continuous sup-semilattice and let B be a CLbasis of L with bottom. One can check that baseMap B is monotone.

The following propositions are true:

(61) Let L be a lower-bounded continuous sup-semilattice and B be a CLbasis of L with bottom. Then idsMap sub(B) is sups-preserving.

(62) For every up-complete non empty poset L and for every non empty full relational substructure S of L holds supMap S = SupMap(L) $\cdot$ idsMap S.

(63) For every lower-bounded continuous sup-semilattice L and for every CLbasis B of L with bottom holds $\langle$supMap sub(B), baseMap $B\rangle$ is Galois.

(64) Let L be a lower-bounded continuous sup-semilattice and B be a CLbasis of L with bottom. Then supMap sub(B) is upper adjoint and baseMap B is lower adjoint.

(65) Let L be a lower-bounded continuous sup-semilattice and B be a CLbasis of L with bottom. Then rng supMap sub(B) = the carrier of L.

(66) Let L be a lower-bounded continuous sup-semilattice and B be a CLbasis of L with bottom. Then supMap sub(B) is infs-preserving and sups-preserving.

(67) Let L be a lower-bounded continuous sup-semilattice and B be a CLbasis of L with bottom. Then baseMap B is sups-preserving.

Let L be a lower-bounded continuous sup-semilattice and let B be a CLbasis of L with bottom. One can verify that supMap sub(B) is infs-preserving and sups-preserving and baseMap B is sups-preserving.

One can prove the following propositions:

(69)[2] Let L be a lower-bounded continuous sup-semilattice and B be a CLbasis of L with bottom. Then the carrier of CompactSublatt($\langle$Ids(sub(B)), $\subseteq \rangle$) $= \{\downarrow b : b$ ranges over elements of sub(B)$\}$.

(70) Let L be a lower-bounded continuous sup-semilattice and B be a CLbasis of L with bottom. Then CompactSublatt($\langle$Ids(sub(B)), $\subseteq\rangle$) and sub(B) are isomorphic.

(71) Let L be a continuous lower-bounded lattice and B be a CLbasis of L with bottom. Suppose that for every CLbasis B_1 of L with bottom holds $B \subseteq B_1$. Let J be an element of $\langle$Ids(sub(B)), $\subseteq\rangle$. Then $J = \Downarrow \bigsqcup_L J \cap B$.

(72) Let L be a continuous lower-bounded lattice. Then L is algebraic if and only if the following conditions are satisfied:

(i) the carrier of CompactSublatt(L) is a CLbasis of L with bottom, and

(ii) for every CLbasis B of L with bottom holds the carrier of CompactSublatt(L) $\subseteq B$.

(73) Let L be a continuous lower-bounded lattice. Then L is algebraic if and only if there exists a CLbasis B of L with bottom such that for every CLbasis B_1 of L with bottom holds $B \subseteq B_1$.

References

[1] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(**5**):719–725, 1991.
[2] Grzegorz Bancerek. Bounds in posets and relational substructures. *Formalized Mathematics*, 6(**1**):81–91, 1997.
[3] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(**1**):93–107, 1997.
[4] Grzegorz Bancerek. The "way-below" relation. *Formalized Mathematics*, 6(**1**):169–176, 1997.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Czesław Byliński. Galois connections. *Formalized Mathematics*, 6(**1**):131–143, 1997.
[7] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, Berlin, Heidelberg, New York, 1980.
[8] Adam Grabowski. On the category of posets. *Formalized Mathematics*, 5(**4**):501–505, 1996.
[9] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Formalized Mathematics*, 6(**1**):117–121, 1997.
[10] Robert Milewski. Algebraic lattices. *Formalized Mathematics*, 6(**2**):249–254, 1997.
[11] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[12] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[13] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[14] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[2]The proposition (68) has been removed.

[15] Mariusz Żynel and Czesław Byliński. Properties of relational structures, posets, lattices and maps. *Formalized Mathematics*, 6(**1**):123–130, 1997.

Received November 28, 1998