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Properties of the Product of Compact Topological Spaces

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The notation and terminology used in this paper are introduced in the following articles: [12], [16], [15], [4], [17], [9], [2], [11], [6], [18], [5], [13], [19], [14], [7], [1], [3], [10], and [8].

1. Preliminaries

One can prove the following proposition

(1) For all topological spaces S, T holds $\Omega_{[: S, T :]} = [: \Omega_S, \Omega_T :]$.

Let X be a set and let Y be an empty set. Note that $[: X, Y :]$ is empty.

Let X be an empty set and let Y be a set. Observe that $[: X, Y :]$ is empty.

We now state the proposition

(2) Let X, Y be non empty topological spaces and x be a point of X. Then $Y \longmapsto x$ is a continuous map from Y into $X \restriction \{x\}$.

Let T be a non empty topological structure. One can verify that id_T is homeomorphism.

Let S, T be non empty topological structures. Let us notice that the predicate S and T are homeomorphic is reflexive and symmetric.

The following proposition is true

(3) Let S, T, V be non empty topological spaces. Suppose S and T are homeomorphic and T and V are homeomorphic. Then S and V are homeomorphic.

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2. On the Projections and Empty Topological Spaces

Let T be a topological structure and let P be an empty subset of the carrier of T. One can verify that $T \restriction P$ is empty.

One can check that there exists a topological space which is strict and empty.

One can prove the following propositions:

(4) For every topological space T_1 and for every empty topological space T_2 holds $[\!: T_1, T_2 :\!]$ is empty and $[\!: T_2, T_1 :\!]$ is empty.

(5) Every empty topological space is compact.

Let us note that every topological space which is empty is also compact.

Let T_1 be a topological space and let T_2 be an empty topological space. Observe that $[\!: T_1, T_2 :\!]$ is empty.

One can prove the following propositions:

(6) Let X, Y be non empty topological spaces, x be a point of X, and f be a map from $[\!: Y, X \restriction \{x\} :\!]$ into Y. If $f = \pi_1((\text{the carrier of } Y) \times \{x\})$, then f is one-to-one.

(7) Let X, Y be non empty topological spaces, x be a point of X, and f be a map from $[\!: X \restriction \{x\}, Y :\!]$ into Y. If $f = \pi_2(\{x\} \times \text{the carrier of } Y)$, then f is one-to-one.

(8) Let X, Y be non empty topological spaces, x be a point of X, and f be a map from $[\!: Y, X \restriction \{x\} :\!]$ into Y. If $f = \pi_1((\text{the carrier of } Y) \times \{x\})$, then $f^{-1} = \langle \mathrm{id}_Y, Y \longmapsto x \rangle$.

(9) Let X, Y be non empty topological spaces, x be a point of X, and f be a map from $[\!: X \restriction \{x\}, Y :\!]$ into Y. If $f = \pi_2(\{x\} \times \text{the carrier of } Y)$, then $f^{-1} = \langle Y \longmapsto x, \mathrm{id}_Y \rangle$.

(10) Let X, Y be non empty topological spaces, x be a point of X, and f be a map from $[\!: Y, X \restriction \{x\} :\!]$ into Y. If $f = \pi_1((\text{the carrier of } Y) \times \{x\})$, then f is a homeomorphism.

(11) Let X, Y be non empty topological spaces, x be a point of X, and f be a map from $[\!: X \restriction \{x\}, Y :\!]$ into Y. If $f = \pi_2(\{x\} \times \text{the carrier of } Y)$, then f is a homeomorphism.

3. On the Product of Compact Spaces

One can prove the following propositions:

(12) Let X be a non empty topological space, Y be a compact non empty topological space, G be an open subset of $[\!: X, Y :\!]$, and x be a set. Suppose $x \in \{x'; x' \text{ ranges over points of } X\colon [\!: \{x'\}, \text{the carrier of } Y :\!] \subseteq G\}$. Then

there exists a many sorted set f indexed by the carrier of Y such that for every set i if $i \in$ the carrier of Y, then there exists a subset G_1 of X and there exists a subset H_1 of Y such that $f(i) = \langle G_1, H_1 \rangle$ and $\langle x, i \rangle \in [\colon G_1, H_1 \colon]$ and G_1 is open and H_1 is open and $[\colon G_1, H_1 \colon] \subseteq G$.

(13) Let X be a non empty topological space, Y be a compact non empty topological space, G be an open subset of $[\colon Y, X \colon]$, and x be a set. Suppose $x \in \{y; y$ ranges over points of X: $[\colon \Omega_Y, \{y\} \colon] \subseteq G\}$. Then there exists an open subset R of X such that $x \in R$ and $R \subseteq \{y; y$ ranges over points of X: $[\colon \Omega_Y, \{y\} \colon] \subseteq G\}$.

(14) Let X be a non empty topological space, Y be a compact non empty topological space, and G be an open subset of $[\colon Y, X \colon]$. Then $\{x; x$ ranges over points of X: $[\colon \Omega_Y, \{x\} \colon] \subseteq G\} \in$ the topology of X.

(15) For all non empty topological spaces X, Y and for every point x of X holds $[\colon X \restriction \{x\}, Y \colon]$ and Y are homeomorphic.

(16) For all non empty topological spaces S, T such that S and T are homeomorphic and S is compact holds T is compact.

(17) For all topological spaces X, Y and for every subspace X_1 of X holds $[\colon Y, X_1 \colon]$ is a subspace of $[\colon Y, X \colon]$.

(18) Let X be a non empty topological space, Y be a compact non empty topological space, x be a point of X, and Z be a subset of $[\colon Y, X \colon]$. If $Z = [\colon \Omega_Y, \{x\} \colon]$, then Z is compact.

(19) Let X be a non empty topological space, Y be a compact non empty topological space, and x be a point of X. Then $[\colon Y, X \restriction \{x\} \colon]$ is compact.

(20) Let X, Y be compact non empty topological spaces and R be a family of subsets of X. Suppose $R = \{Q; Q$ ranges over open subsets of X: $[\colon \Omega_Y, Q \colon] \subseteq \bigcup \mathrm{BaseAppr}(\Omega_{[\colon Y, X \colon]})\}$. Then R is open and a cover of Ω_X.

(21) Let X, Y be compact non empty topological spaces, R be a family of subsets of X, and F be a family of subsets of $[\colon Y, X \colon]$. Suppose that

(i) F is a cover of $[\colon Y, X \colon]$ and open, and

(ii) $R = \{Q; Q$ ranges over open subsets of X: $\bigvee_{F_1 \colon \text{family of subsets of } [\colon Y, X \colon]} (F_1 \subseteq F \wedge F_1 \text{ is finite} \wedge [\colon \Omega_Y, Q \colon] \subseteq \bigcup F_1)\}$.

Then R is open and a cover of X.

(22) Let X, Y be compact non empty topological spaces, R be a family of subsets of X, and F be a family of subsets of $[\colon Y, X \colon]$. Suppose that

(i) F is a cover of $[\colon Y, X \colon]$ and open, and

(ii) $R = \{Q; Q$ ranges over open subsets of X: $\bigvee_{F_1 \colon \text{family of subsets of } [\colon Y, X \colon]} (F_1 \subseteq F \wedge F_1 \text{ is finite} \wedge [\colon \Omega_Y, Q \colon] \subseteq \bigcup F_1)\}$.

Then there exists a family C of subsets of X such that $C \subseteq R$ and C is finite and a cover of X.

(23) Let X, Y be compact non empty topological spaces and F be a family of

subsets of $[: Y, X :]$. Suppose F is a cover of $[: Y, X :]$ and open. Then there exists a family G of subsets of $[: Y, X :]$ such that $G \subseteq F$ and G is a cover of $[: Y, X :]$ and finite.

(24) For all topological spaces T_1, T_2 such that T_1 is compact and T_2 is compact holds $[: T_1, T_2 :]$ is compact.

Let T_1, T_2 be compact topological spaces. Observe that $[: T_1, T_2 :]$ is compact. Next we state two propositions:

(25) Let X, Y be non empty topological spaces, X_1 be a non empty subspace of X, and Y_1 be a non empty subspace of Y. Then $[: X_1, Y_1 :]$ is a subspace of $[: X, Y :]$.

(26) Let X, Y be non empty topological spaces, Z be a non empty subset of $[: Y, X :]$, V be a non empty subset of X, and W be a non empty subset of Y. Suppose $Z = [: W, V :]$. Then the topological structure of $[: Y \restriction W, X \restriction V :] =$ the topological structure of $[: Y, X :] \restriction Z$.

Let T be a topological space. Observe that there exists a subset of T which is compact.

Let T be a topological space and let P be a compact subset of T. Note that $T \restriction P$ is compact.

We now state the proposition

(27) Let T_1, T_2 be topological spaces, S_1 be a subset of T_1, and S_2 be a subset of T_2. If S_1 is compact and S_2 is compact, then $[: S_1, S_2 :]$ is a compact subset of $[: T_1, T_2 :]$.

References

[1] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Introduction to categories and functors. *Formalized Mathematics*, 1(**2**):409–420, 1990.
[4] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[5] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.
[6] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[7] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[8] Michał Muzalewski. Categories of groups. *Formalized Mathematics*, 2(**4**):563–571, 1991.
[9] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[10] Beata Padlewska. Locally connected spaces. *Formalized Mathematics*, 2(**1**):93–96, 1991.
[11] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[13] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[14] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[16] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[17] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[18] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

[19] Mariusz Żynel and Adam Guzowski. T_0 topological spaces. *Formalized Mathematics*, 5(**1**):75–77, 1996.

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