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Logic Gates and Logical Equivalence of Adders

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Summary. This is an experimental article which shows that logical correctness of logic circuits can be easily proven by the Mizar system. First, we define the notion of logic gates. Then we prove that an MSB carry of '4 Bit Carry Skip Adder' is equivalent to an MSB carry of a normal 4 bit adder. In the last theorem, we show that outputs of the '4 Bit Carry Look Ahead Adder' are equivalent to the corresponding outputs of the normal 4 bits adder. The policy here is as follows: when the functional (semantic) correctness of a system is already proven, and the correspondence of the system to a (normal) logic circuit is given, it is enough to prove the correctness of the new circuit if we only prove the logical equivalence between them. Although the article is very fundamental (it contains few environment files), it can be applied to real problems. The key of the method introduced here is to put the specification of the logic circuit into the Mizar propositional formulae, and to use the strong inference ability of the Mizar checker. The proof is done formally so that the automation of the proof writing is possible. Even in the 5.3.07 version of Mizar, it can handle a formulae of more than 100 lines, and a formula which contains more than 100 variables. This means that the Mizar system is enough to prove logical correctness of middle scaled logic circuits.

MML Identifier: `GATE_1`.

The articles [2] and [1] provide the terminology and notation for this paper.

1. Definition of Logical Values and Logic Gates

Let a be a set. We introduce NE a as an antonym of a is empty.

We now state three propositions:

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(1) For every set a such that $a = \{\emptyset\}$ holds NE a.

(2) There exists a set a such that NE a.

(3) NE $\emptyset$ iff *contradiction*.

let a be a set. The functor NOT1 a yielding a set is defined by:

(Def. 1) $\text{NOT1}\, a = \begin{cases} \emptyset, & \text{if NE } a, \\ \{\emptyset\}, & \text{otherwise.} \end{cases}$

The following proposition is true

(4) For every set a holds NE NOT1 a iff not NE a.

In the sequel a, b are sets.

We now state the proposition

(5) NE NOT1 $\emptyset$.

Let a, b be sets. The functor AND2(a, b) yields a set and is defined by:

(Def. 2) $\text{AND2}(a, b) = \begin{cases} \text{NOT1}\, \emptyset, & \text{if NE } a \text{ and NE } b, \\ \emptyset, & \text{otherwise.} \end{cases}$

Next we state the proposition

(6) For all sets a, b holds NE AND2(a, b) iff NE a and NE b.

Let a, b be sets. The functor OR2(a, b) yielding a set is defined as follows:

(Def. 3) $\text{OR2}(a, b) = \begin{cases} \text{NOT1}\, \emptyset, & \text{if NE } a \text{ or NE } b, \\ \emptyset, & \text{otherwise.} \end{cases}$

Next we state the proposition

(7) For all sets a, b holds NE OR2(a, b) iff NE a or NE b.

Let a, b be sets. The functor XOR2(a, b) yields a set and is defined by:

(Def. 4) $\text{XOR2}(a, b) = \begin{cases} \text{NOT1}\, \emptyset, & \text{if NE } a \text{ and not NE } b \text{ or not NE } a \text{ and NE } b, \\ \emptyset, & \text{otherwise.} \end{cases}$

The following four propositions are true:

(8) For all sets a, b holds NE XOR2(a, b) iff NE a and not NE b or not NE a and NE b.

(9) NE XOR2(a, a) iff *contradiction*.

(10) NE XOR2$(a, \emptyset)$ iff NE a.

(11) NE XOR2(a, b) iff NE XOR2(b, a).

Let a, b be sets. The functor EQV2(a, b) yielding a set is defined by:

(Def. 5) $\text{EQV2}(a, b) = \begin{cases} \text{NOT1}\, \emptyset, & \text{if NE } a \text{ iff NE } b, \\ \emptyset, & \text{otherwise.} \end{cases}$

We now state two propositions:

(12) For all sets a, b holds NE EQV2(a, b) iff NE a iff NE b.

(13) NE EQV2(a, b) iff not NE XOR2(a, b).

Let a, b be sets. The functor NAND2(a, b) yielding a set is defined by:

(Def. 6) $\text{NAND2}(a,b) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if not NE } a \text{ or not NE } b, \\ \emptyset, & \text{otherwise.} \end{cases}$

One can prove the following proposition

(14) For all sets a, b holds NE NAND2(a,b) iff not NE a or not NE b.

Let a, b be sets. The functor NOR2(a,b) yielding a set is defined as follows:

(Def. 7) $\text{NOR2}(a,b) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if not NE } a \text{ and not NE } b, \\ \emptyset, & \text{otherwise.} \end{cases}$

We now state the proposition

(15) For all sets a, b holds NE NOR2(a,b) iff not NE a and not NE b.

Let a, b, c be sets. The functor AND3(a,b,c) yields a set and is defined by:

(Def. 8) $\text{AND3}(a,b,c) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if NE } a \text{ and NE } b \text{ and NE } c, \\ \emptyset, & \text{otherwise.} \end{cases}$

One can prove the following proposition

(16) For all sets a, b, c holds NE AND3(a,b,c) iff NE a and NE b and NE c.

Let a, b, c be sets. The functor OR3(a,b,c) yielding a set is defined by:

(Def. 9) $\text{OR3}(a,b,c) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if NE } a \text{ or NE } b \text{ or NE } c, \\ \emptyset, & \text{otherwise.} \end{cases}$

One can prove the following proposition

(17) For all sets a, b, c holds NE OR3(a,b,c) iff NE a or NE b or NE c.

Let a, b, c be sets. The functor XOR3(a,b,c) yielding a set is defined by:

(Def. 10) $\text{XOR3}(a,b,c) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if NE } a \text{ and not NE } b \text{ or not NE } a \text{ and NE } b \text{ but not NE } c \text{ or not NE } a \text{ or not NE } b \text{ but not NE } a \text{ or not NE } b \text{ and NE } c, \\ \emptyset, & \text{otherwise.} \end{cases}$

We now state the proposition

(18) Let a, b, c be sets. Then NE XOR3(a,b,c) if and only if one of the following conditions is satisfied:

(i) NE a and not NE b or not NE a and NE b but not NE c, or

(ii) not NE a or not NE b but not NE a or not NE b and NE c.

Let a, b, c be sets. The functor MAJ3(a,b,c) yields a set and is defined as follows:

(Def. 11) $\text{MAJ3}(a,b,c) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if NE } a \text{ and NE } b \text{ or NE } b \text{ and NE } c \text{ or NE } c \text{ and NE } a, \\ \emptyset, & \text{otherwise.} \end{cases}$

The following proposition is true

(19) For all sets a, b, c holds NE MAJ3(a,b,c) iff NE a and NE b or NE b and NE c or NE c and NE a.

Let a, b, c be sets. The functor NAND3(a,b,c) yielding a set is defined by:

(Def. 12) $\text{NAND3}(a,b,c) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if not NE } a \text{ or not NE } b \text{ or not NE } c, \\ \emptyset, & \text{otherwise.} \end{cases}$

The following proposition is true

(20) For all sets a, b, c holds NE NAND3(a,b,c) iff not NE a or not NE b or not NE c.

Let a, b, c be sets. The functor NOR3(a,b,c) yields a set and is defined by:

(Def. 13) $\text{NOR3}(a,b,c) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if not NE } a \text{ and not NE } b \text{ and not NE } c, \\ \emptyset, & \text{otherwise.} \end{cases}$

We now state the proposition

(21) For all sets a, b, c holds NE NOR3(a,b,c) iff not NE a and not NE b and not NE c.

Let a, b, c, d be sets. The functor AND4(a,b,c,d) yields a set and is defined by:

(Def. 14) $\text{AND4}(a,b,c,d) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if NE } a \text{ and NE } b \text{ and NE } c \text{ and NE } d, \\ \emptyset, & \text{otherwise.} \end{cases}$

One can prove the following proposition

(22) For all sets a, b, c, d holds NE AND4(a,b,c,d) iff NE a and NE b and NE c and NE d.

Let a, b, c, d be sets. The functor OR4(a,b,c,d) yielding a set is defined as follows:

(Def. 15) $\text{OR4}(a,b,c,d) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if NE } a \text{ or NE } b \text{ or NE } c \text{ or NE } d, \\ \emptyset, & \text{otherwise.} \end{cases}$

The following proposition is true

(23) For all sets a, b, c, d holds NE OR4(a,b,c,d) iff NE a or NE b or NE c or NE d.

Let a, b, c, d be sets. The functor NAND4(a,b,c,d) yielding a set is defined by:

(Def. 16) $\text{NAND4}(a,b,c,d) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if not NE } a \text{ or not NE } b \text{ or not NE } c \text{ or not NE } d, \\ \emptyset, & \text{otherwise.} \end{cases}$

Next we state the proposition

(24) For all sets a, b, c, d holds NE NAND4(a,b,c,d) iff not NE a or not NE b or not NE c or not NE d.

Let a, b, c, d be sets. The functor NOR4(a,b,c,d) yielding a set is defined by:

(Def. 17) $\text{NOR4}(a,b,c,d) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if not NE } a \text{ and not NE } b \text{ and not NE } c \text{ and not NE } d, \\ \emptyset, & \text{otherwise.} \end{cases}$

The following proposition is true

(25) For all sets a, b, c, d holds NE NOR4(a, b, c, d) iff not NE a and not NE b and not NE c and not NE d.

Let a, b, c, d, e be sets. The functor AND5(a, b, c, d, e) yielding a set is defined as follows:

(Def. 18) $\text{AND5}(a, b, c, d, e) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if NE } a \text{ and NE } b \text{ and NE } c \text{ and NE } d \\ & \quad \text{and NE } e, \\ \emptyset, & \text{otherwise.} \end{cases}$

Next we state the proposition

(26) For all sets a, b, c, d, e holds NE AND5(a, b, c, d, e) iff NE a and NE b and NE c and NE d and NE e.

Let a, b, c, d, e be sets. The functor OR5(a, b, c, d, e) yields a set and is defined by:

(Def. 19) $\text{OR5}(a, b, c, d, e) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if NE } a \text{ or NE } b \text{ or NE } c \text{ or NE } d \text{ or NE } e, \\ \emptyset, & \text{otherwise.} \end{cases}$

The following proposition is true

(27) For all sets a, b, c, d, e holds NE OR5(a, b, c, d, e) iff NE a or NE b or NE c or NE d or NE e.

Let a, b, c, d, e be sets. The functor NAND5(a, b, c, d, e) yields a set and is defined as follows:

(Def. 20) $\text{NAND5}(a, b, c, d, e) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if not NE } a \text{ or not NE } b \text{ or not NE } c \\ & \quad \text{or not NE } d \text{ or not NE } e, \\ \emptyset, & \text{otherwise.} \end{cases}$

The following proposition is true

(28) For all sets a, b, c, d, e holds NE NAND5(a, b, c, d, e) iff not NE a or not NE b or not NE c or not NE d or not NE e.

Let a, b, c, d, e be sets. The functor NOR5(a, b, c, d, e) yielding a set is defined as follows:

(Def. 21) $\text{NOR5}(a, b, c, d, e) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if not NE } a \text{ and not NE } b \text{ and not NE } c \\ & \quad \text{and not NE } d \text{ and not NE } e, \\ \emptyset, & \text{otherwise.} \end{cases}$

We now state the proposition

(29) For all sets a, b, c, d, e holds NE NOR5(a, b, c, d, e) iff not NE a and not NE b and not NE c and not NE d and not NE e.

Let a, b, c, d, e, f be sets. The functor AND6(a, b, c, d, e, f) yielding a set is defined by:

(Def. 22) $\text{AND6}(a, b, c, d, e, f) = \begin{cases} \text{NOT1}\,\emptyset, & \text{if NE } a \text{ and NE } b \text{ and NE } c \text{ and NE } d \\ & \quad \text{and NE } e \text{ and NE } f, \\ \emptyset, & \text{otherwise.} \end{cases}$

Next we state the proposition

(30) Let a, b, c, d, e, f be sets. Then NE AND6(a, b, c, d, e, f) if and only if the following conditions are satisfied:

(i) NE a,
(ii) NE b,
(iii) NE c,
(iv) NE d,
(v) NE e, and
(vi) NE f.

Let a, b, c, d, e, f be sets. The functor OR6(a, b, c, d, e, f) yielding a set is defined by:

(Def. 23) $$\mathrm{OR6}(a,b,c,d,e,f) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ or NE } b \text{ or NE } c \text{ or NE } d \text{ or NE } e \text{ or NE } f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The following proposition is true

(31) Let a, b, c, d, e, f be sets. Then NE OR6(a, b, c, d, e, f) if and only if one of the following conditions is satisfied:

(i) NE a, or
(ii) NE b, or
(iii) NE c, or
(iv) NE d, or
(v) NE e, or
(vi) NE f.

Let a, b, c, d, e, f be sets. The functor NAND6(a, b, c, d, e, f) yields a set and is defined by:

(Def. 24) $$\mathrm{NAND6}(a,b,c,d,e,f) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ or not NE } b \text{ or not NE } c \text{ or not NE } d \text{ or not NE } e \text{ or not NE } f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The following proposition is true

(32) Let a, b, c, d, e, f be sets. Then NE NAND6(a, b, c, d, e, f) if and only if one of the following conditions is satisfied:

(i) not NE a, or
(ii) not NE b, or
(iii) not NE c, or
(iv) not NE d, or
(v) not NE e, or
(vi) not NE f.

Let a, b, c, d, e, f be sets. The functor NOR6(a, b, c, d, e, f) yields a set and is defined as follows:

(Def. 25) $$\mathrm{NOR6}(a,b,c,d,e,f) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ and not NE } b \text{ and not NE } c \text{ and not NE } d \text{ and not NE } e \text{ and not NE } f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

One can prove the following proposition

(33) Let a, b, c, d, e, f be sets. Then NE NOR6(a, b, c, d, e, f) if and only if the following conditions are satisfied:

(i) not NE a,

(ii) not NE b,

(iii) not NE c,

(iv) not NE d,

(v) not NE e, and

(vi) not NE f.

Let a, b, c, d, e, f, g be sets. The functor AND7(a, b, c, d, e, f, g) yields a set and is defined by:

$$\text{(Def. 26)}\quad \mathrm{AND7}(a,b,c,d,e,f,g) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ and NE } b \text{ and NE } c \text{ and} \\ & \text{NE } d \text{ and NE } e \text{ and NE } f \text{ and NE } g, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Next we state the proposition

(34) Let a, b, c, d, e, f, g be sets. Then NE AND7(a, b, c, d, e, f, g) if and only if the following conditions are satisfied:
NE a and NE b and NE c and NE d and NE e and NE f and NE g.

Let a, b, c, d, e, f, g be sets. The functor OR7(a, b, c, d, e, f, g) yielding a set is defined as follows:

$$\text{(Def. 27)}\quad \mathrm{OR7}(a,b,c,d,e,f,g) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ or NE } b \text{ or NE } c \text{ or NE } d \text{ or} \\ & \text{NE } e \text{ or NE } f \text{ or NE } g, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Next we state the proposition

(35) Let a, b, c, d, e, f, g be sets. Then NE OR7(a, b, c, d, e, f, g) if and only if one of the following conditions is satisfied:
NE a or NE b or NE c or NE d or NE e or NE f or NE g.

Let a, b, c, d, e, f, g be sets. The functor NAND7(a, b, c, d, e, f, g) yielding a set is defined as follows:

$$\text{(Def. 28)}\quad \mathrm{NAND7}(a,b,c,d,e,f,g) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ or not NE } b \text{ or} \\ & \text{not NE } c \text{ or not NE } d \text{ or not NE } e \text{ or not} \\ & \text{NE } f \text{ or not NE } g, \\ \emptyset, & \text{otherwise.} \end{cases}$$

One can prove the following proposition

(36) Let a, b, c, d, e, f, g be sets. Then NE NAND7(a, b, c, d, e, f, g) if and only if one of the following conditions is satisfied:
not NE a or not NE b or not NE c or not NE d or not NE e or not NE f or not NE g.

Let a, b, c, d, e, f, g be sets. The functor NOR7(a, b, c, d, e, f, g) yielding a set is defined as follows:

(Def. 29) $$\mathrm{NOR7}(a,b,c,d,e,f,g) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ and not NE } b \text{ and not NE } c \text{ and not NE } d \text{ and not NE } e \text{ and not NE } f \text{ and not NE } g, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Next we state the proposition

(37) Let a, b, c, d, e, f, g be sets. Then NE $\mathrm{NOR7}(a,b,c,d,e,f,g)$ if and only if the following conditions are satisfied:
not NE a and not NE b and not NE c and not NE d and not NE e and not NE f and not NE g.

Let a, b, c, d, e, f, g, h be sets. The functor $\mathrm{AND8}(a,b,c,d,e,f,g,h)$ yields a set and is defined by:

(Def. 30) $$\mathrm{AND8}(a,b,c,d,e,f,g,h) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ and NE } b \text{ and NE } c \text{ and NE } d \text{ and NE } e \text{ and NE } f \text{ and NE } g \text{ and NE } h, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The following proposition is true

(38) Let a, b, c, d, e, f, g, h be sets. Then NE $\mathrm{AND8}(a,b,c,d,e,f,g,h)$ if and only if the following conditions are satisfied:
NE a and NE b and NE c and NE d and NE e and NE f and NE g and NE h.

Let a, b, c, d, e, f, g, h be sets. The functor $\mathrm{OR8}(a,b,c,d,e,f,g,h)$ yielding a set is defined as follows:

(Def. 31) $$\mathrm{OR8}(a,b,c,d,e,f,g,h) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ or NE } b \text{ or NE } c \text{ or NE } d \text{ or NE } e \text{ or NE } f \text{ or NE } g \text{ or NE } h, \\ \emptyset, & \text{otherwise.} \end{cases}$$

One can prove the following proposition

(39) Let a, b, c, d, e, f, g, h be sets. Then NE $\mathrm{OR8}(a,b,c,d,e,f,g,h)$ if and only if one of the following conditions is satisfied:
NE a or NE b or NE c or NE d or NE e or NE f or NE g or NE h.

Let a, b, c, d, e, f, g, h be sets. The functor $\mathrm{NAND8}(a,b,c,d,e,f,g,h)$ yielding a set is defined as follows:

(Def. 32) $$\mathrm{NAND8}(a,b,c,d,e,f,g,h) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ or not NE } b \text{ or not NE } c \text{ or not NE } d \text{ or not NE } e \text{ or not NE } f \text{ or not NE } g \text{ or not NE } h, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Next we state the proposition

(40) Let a, b, c, d, e, f, g, h be sets. Then NE $\mathrm{NAND8}(a,b,c,d,e,f,g,h)$ if and only if one of the following conditions is satisfied:
not NE a or not NE b or not NE c or not NE d or not NE e or not NE f or not NE g or not NE h.

Let a, b, c, d, e, f, g, h be sets. The functor $\mathrm{NOR8}(a,b,c,d,e,f,g,h)$ yielding a set is defined as follows:

(Def. 33) $\mathrm{NOR8}(a,b,c,d,e,f,g,h) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ and not NE } b \text{ and not NE } c \text{ and not NE } d \text{ and not NE } e \text{ and not NE } f \text{ and not NE } g \text{ and not NE } h, \\ \emptyset, & \text{otherwise.} \end{cases}$

One can prove the following proposition

(41) Let a, b, c, d, e, f, g, h be sets. Then NE $\mathrm{NOR8}(a,b,c,d,e,f,g,h)$ if and only if the following conditions are satisfied:
not NE a and not NE b and not NE c and not NE d and not NE e and not NE f and not NE g and not NE h.

2. Logical Equivalence of 4 Bits Adders

We now state the proposition

(42) Let c_1, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, c_2, c_3, c_4, c_5, n_1, n_2, n_3, n_4, n, c_6 be sets. Suppose that
NE c_2 iff NE $\mathrm{MAJ3}(x_1,y_1,c_1)$ and NE c_3 iff NE $\mathrm{MAJ3}(x_2,y_2,c_2)$ and NE c_4 iff NE $\mathrm{MAJ3}(x_3,y_3,c_3)$ and NE c_5 iff NE $\mathrm{MAJ3}(x_4,y_4,c_4)$ and NE n_1 iff NE $\mathrm{OR2}(x_1,y_1)$ and NE n_2 iff NE $\mathrm{OR2}(x_2,y_2)$ and NE n_3 iff NE $\mathrm{OR2}(x_3,y_3)$ and NE n_4 iff NE $\mathrm{OR2}(x_4,y_4)$ and NE n iff NE $\mathrm{AND5}(c_1,n_1,n_2,n_3,n_4)$ and NE c_6 iff NE $\mathrm{OR2}(c_5,n)$. Then NE c_5 if and only if NE c_6.

Let a, b be sets. The functor $\mathrm{MODADD2}(a,b)$ yields a set and is defined as follows:

(Def. 34) $\mathrm{MODADD2}(a,b) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ or NE } b \text{ but NE } a \text{ but NE } b, \\ \emptyset, & \text{otherwise.} \end{cases}$

Next we state the proposition

(43) For all sets a, b holds NE $\mathrm{MODADD2}(a,b)$ iff NE a or NE b but NE a but NE b.

Let a, b, c be sets. The functor $\mathrm{ADD1}(a,b,c)$ yields a set and is defined by:

(Def. 35) $\mathrm{ADD1}(a,b,c) = \mathrm{XOR3}(a,b,c)$.

Let a, b, c be sets. The functor $\mathrm{CARR1}(a,b,c)$ yielding a set is defined by:

(Def. 36) $\mathrm{CARR1}(a,b,c) = \mathrm{MAJ3}(a,b,c)$.

Let a_1, b_1, a_2, b_2, c be sets. The functor $\mathrm{ADD2}(a_2,b_2,a_1,b_1,c)$ yielding a set is defined as follows:

(Def. 37) $\mathrm{ADD2}(a_2,b_2,a_1,b_1,c) = \mathrm{XOR3}(a_2,b_2,\mathrm{CARR1}(a_1,b_1,c))$.

Let a_1, b_1, a_2, b_2, c be sets. The functor $\mathrm{CARR2}(a_2, b_2, a_1, b_1, c)$ yields a set and is defined as follows:

(Def. 38) $\mathrm{CARR2}(a_2, b_2, a_1, b_1, c) = \mathrm{MAJ3}(a_2, b_2, \mathrm{CARR1}(a_1, b_1, c))$.

Let a_1, b_1, a_2, b_2, a_3, b_3, c be sets. The functor $\mathrm{ADD3}(a_3, b_3, a_2, b_2, a_1, b_1, c)$ yields a set and is defined by:

(Def. 39) $\mathrm{ADD3}(a_3, b_3, a_2, b_2, a_1, b_1, c) = \mathrm{XOR3}(a_3, b_3, \mathrm{CARR2}(a_2, b_2, a_1, b_1, c))$.

Let a_1, b_1, a_2, b_2, a_3, b_3, c be sets. The functor $\mathrm{CARR3}(a_3, b_3, a_2, b_2, a_1, b_1, c)$ yields a set and is defined as follows:

(Def. 40) $\mathrm{CARR3}(a_3, b_3, a_2, b_2, a_1, b_1, c) = \mathrm{MAJ3}(a_3, b_3, \mathrm{CARR2}(a_2, b_2, a_1, b_1, c))$.

Let a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, c be sets.

The functor $\mathrm{ADD4}(a_4, b_4, a_3, b_3, a_2, b_2, a_1, b_1, c)$ yielding a set is defined by:

(Def. 41) $\mathrm{ADD4}(a_4, b_4, a_3, b_3, a_2, b_2, a_1, b_1, c) = \mathrm{XOR3}(a_4, b_4, \mathrm{CARR3}(a_3, b_3, a_2, b_2, a_1, b_1, c))$.

Let a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, c be sets.

The functor $\mathrm{CARR4}(a_4, b_4, a_3, b_3, a_2, b_2, a_1, b_1, c)$ yields a set and is defined as follows:

(Def. 42) $\mathrm{CARR4}(a_4, b_4, a_3, b_3, a_2, b_2, a_1, b_1, c) = \mathrm{MAJ3}(a_4, b_4, \mathrm{CARR3}(a_3, b_3, a_2, b_2, a_1, b_1, c))$.

One can prove the following proposition

(44) Let c_1, x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, c_4, q_1, p_1, s_1, q_2, p_2, s_2, q_3, p_3, s_3, q_4, p_4, s_4, c_7, c_8, l_2, t_2, l_3, m_3, t_3, l_4, m_4, n_4, t_4, l_5, m_5, n_5, o_5, s_5, s_6, s_7, s_8 be sets such that NE q_1 iff NE $\mathrm{NOR2}(x_1, y_1)$ and NE p_1 iff NE $\mathrm{NAND2}(x_1, y_1)$ and NE s_1 iff NE $\mathrm{MODADD2}(x_1, y_1)$ and NE q_2 iff NE $\mathrm{NOR2}(x_2, y_2)$ and NE p_2 iff NE $\mathrm{NAND2}(x_2, y_2)$ and NE s_2 iff NE $\mathrm{MODADD2}(x_2, y_2)$ and NE q_3 iff NE $\mathrm{NOR2}(x_3, y_3)$ and NE p_3 iff NE $\mathrm{NAND2}(x_3, y_3)$ and NE s_3 iff NE $\mathrm{MODADD2}(x_3, y_3)$ and NE q_4 iff NE $\mathrm{NOR2}(x_4, y_4)$ and NE p_4 iff NE $\mathrm{NAND2}(x_4, y_4)$ and NE s_4 iff NE $\mathrm{MODADD2}(x_4, y_4)$ and NE c_7 iff NE NOT1 c_1 and NE c_8 iff NE NOT1 c_7 and NE s_5 iff NE $\mathrm{XOR2}(c_8, s_1)$ and NE l_2 iff NE $\mathrm{AND2}(c_7, p_1)$ and NE t_2 iff NE $\mathrm{NOR2}(l_2, q_1)$ and NE s_6 iff NE $\mathrm{XOR2}(t_2, s_2)$ and NE l_3 iff NE $\mathrm{AND2}(q_1, p_2)$ and NE m_3 iff NE $\mathrm{AND3}(p_2, p_1, c_7)$ and NE t_3 iff NE $\mathrm{NOR3}(l_3, m_3, q_2)$ and NE s_7 iff NE $\mathrm{XOR2}(t_3, s_3)$ and NE l_4 iff NE $\mathrm{AND2}(q_2, p_3)$ and NE m_4 iff NE $\mathrm{AND3}(q_1, p_3, p_2)$ and NE n_4 iff NE $\mathrm{AND4}(p_3, p_2, p_1, c_7)$ and NE t_4 iff NE $\mathrm{NOR4}(l_4, m_4, n_4, q_3)$ and NE s_8 iff NE $\mathrm{XOR2}(t_4, s_4)$ and NE l_5 iff NE $\mathrm{AND2}(q_3, p_4)$ and NE m_5 iff NE $\mathrm{AND3}(q_2, p_4, p_3)$ and NE n_5 iff NE $\mathrm{AND4}(q_1, p_4, p_3, p_2)$ and NE o_5 iff NE $\mathrm{AND5}(p_4, p_3, p_2, p_1, c_7)$ and NE c_4 iff NE $\mathrm{NOR5}(q_4, l_5, m_5, n_5, o_5)$. Then

(i) NE s_5 iff NE $\mathrm{ADD1}(x_1, y_1, c_1)$,

(ii) NE s_6 iff NE $\mathrm{ADD2}(x_2, y_2, x_1, y_1, c_1)$,

(iii) NE s_7 iff NE $\mathrm{ADD3}(x_3, y_3, x_2, y_2, x_1, y_1, c_1)$,

(iv) NE s_8 iff NE ADD4$(x_4, y_4, x_3, y_3, x_2, y_2, x_1, y_1, c_1)$, and
(v) NE c_4 iff NE CARR4$(x_4, y_4, x_3, y_3, x_2, y_2, x_1, y_1, c_1)$.

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