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Hilbert Positive Propositional Calculus

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The papers [4], [5], [3], [1], and [2] provide the notation and terminology for this paper.

1. Definition of the Language

Let D be a set. We say that D has VERUM if and only if:

(Def. 1) $\langle 0 \rangle \in D$.

Let D be a set. We say that D has implication if and only if:

(Def. 2) For all finite sequences p, q such that $p \in D$ and $q \in D$ holds $\langle 1 \rangle ^\frown p ^\frown q \in D$.

Let D be a set. We say that D has conjunction if and only if:

(Def. 3) For all finite sequences p, q such that $p \in D$ and $q \in D$ holds $\langle 2 \rangle ^\frown p ^\frown q \in D$.

Let D be a set. We say that D has propositional variables if and only if:

(Def. 4) For every natural number n holds $\langle 3 + n \rangle \in D$.

Let D be a set. We say that D is HP-closed if and only if:

(Def. 5) $D \subseteq \mathbb{N}^*$ and D has VERUM, implication, conjunction, and propositional variables.

Let us note that every set which is HP-closed is also non empty and has VERUM, implication, conjunction, and propositional variables and every subset of $\mathbb{N}^*$ which has VERUM, implication, conjunction, and propositional variables is HP-closed.

The set HP-WFF is defined as follows:

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(Def. 6) HP-WFF is HP-closed and for every set D such that D is HP-closed holds HP-WFF $\subseteq D$.

Let us note that HP-WFF is HP-closed.

Let us mention that there exists a set which is HP-closed and non empty.

One can verify that every element of HP-WFF is relation-like and function-like.

Let us mention that every element of HP-WFF is finite sequence-like.

A HP-formula is an element of HP-WFF.

The HP-formula VERUM is defined by:

(Def. 7) VERUM $= \langle 0 \rangle$.

Let p, q be elements of HP-WFF. The functor $p \Rightarrow q$ yielding a HP-formula is defined by:

(Def. 8) $p \Rightarrow q = \langle 1 \rangle \frown p \frown q$.

The functor $p \wedge q$ yielding a HP-formula is defined as follows:

(Def. 9) $p \wedge q = \langle 2 \rangle \frown p \frown q$.

We follow the rules: T, X, Y denote subsets of HP-WFF and p, q, r, s denote elements of HP-WFF.

Let T be a subset of HP-WFF. We say that T is Hilbert theory if and only if the conditions (Def. 10) are satisfied.

(Def. 10)(i) VERUM $\in T$, and

(ii) for all elements p, q, r of HP-WFF holds $p \Rightarrow (q \Rightarrow p) \in T$ and $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in T$ and $p \wedge q \Rightarrow p \in T$ and $p \wedge q \Rightarrow q \in T$ and $p \Rightarrow (q \Rightarrow p \wedge q) \in T$ and if $p \in T$ and $p \Rightarrow q \in T$, then $q \in T$.

Let us consider X. The functor $\mathrm{CnPos}\, X$ yields a subset of HP-WFF and is defined by:

(Def. 11) $r \in \mathrm{CnPos}\, X$ iff for every T such that T is Hilbert theory and $X \subseteq T$ holds $r \in T$.

The subset HP_TAUT of HP-WFF is defined by:

(Def. 12) HP_TAUT $= \mathrm{CnPos}\, \emptyset_{\text{HP-WFF}}$.

The following propositions are true:

(1) VERUM $\in \mathrm{CnPos}\, X$.

(2) $p \Rightarrow (q \Rightarrow p \wedge q) \in \mathrm{CnPos}\, X$.

(3) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in \mathrm{CnPos}\, X$.

(4) $p \Rightarrow (q \Rightarrow p) \in \mathrm{CnPos}\, X$.

(5) $p \wedge q \Rightarrow p \in \mathrm{CnPos}\, X$.

(6) $p \wedge q \Rightarrow q \in \mathrm{CnPos}\, X$.

(7) If $p \in \mathrm{CnPos}\, X$ and $p \Rightarrow q \in \mathrm{CnPos}\, X$, then $q \in \mathrm{CnPos}\, X$.

(8) If T is Hilbert theory and $X \subseteq T$, then $\mathrm{CnPos}\, X \subseteq T$.

(9) $X \subseteq \mathrm{CnPos}\, X$.

(10) If $X \subseteq Y$, then $\mathrm{CnPos}\, X \subseteq \mathrm{CnPos}\, Y$.

(11) $\mathrm{CnPos}\,\mathrm{CnPos}\, X = \mathrm{CnPos}\, X$.

Let X be a subset of HP-WFF. One can verify that $\mathrm{CnPos}\, X$ is Hilbert theory.

We now state two propositions:

(12) T is Hilbert theory iff $\mathrm{CnPos}\, T = T$.

(13) If T is Hilbert theory, then $\mathrm{HP\_TAUT} \subseteq T$.

Let us mention that HP_TAUT is Hilbert theory.

2. The Tautologies of the Hilbert Calculus - Implicational Part

We now state a number of propositions:

(14) $p \Rightarrow p \in \mathrm{HP\_TAUT}$.

(15) If $q \in \mathrm{HP\_TAUT}$, then $p \Rightarrow q \in \mathrm{HP\_TAUT}$.

(16) $p \Rightarrow \mathrm{VERUM} \in \mathrm{HP\_TAUT}$.

(17) $(p \Rightarrow q) \Rightarrow (p \Rightarrow p) \in \mathrm{HP\_TAUT}$.

(18) $(q \Rightarrow p) \Rightarrow (p \Rightarrow p) \in \mathrm{HP\_TAUT}$.

(19) $(q \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in \mathrm{HP\_TAUT}$.

(20) If $p \Rightarrow (q \Rightarrow r) \in \mathrm{HP\_TAUT}$, then $q \Rightarrow (p \Rightarrow r) \in \mathrm{HP\_TAUT}$.

(21) $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r)) \in \mathrm{HP\_TAUT}$.

(22) If $p \Rightarrow q \in \mathrm{HP\_TAUT}$, then $(q \Rightarrow r) \Rightarrow (p \Rightarrow r) \in \mathrm{HP\_TAUT}$.

(23) If $p \Rightarrow q \in \mathrm{HP\_TAUT}$ and $q \Rightarrow r \in \mathrm{HP\_TAUT}$, then $p \Rightarrow r \in \mathrm{HP\_TAUT}$.

(24) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((s \Rightarrow q) \Rightarrow (p \Rightarrow (s \Rightarrow r))) \in \mathrm{HP\_TAUT}$.

(25) $((p \Rightarrow q) \Rightarrow r) \Rightarrow (q \Rightarrow r) \in \mathrm{HP\_TAUT}$.

(26) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow (q \Rightarrow (p \Rightarrow r)) \in \mathrm{HP\_TAUT}$.

(27) $(p \Rightarrow (p \Rightarrow q)) \Rightarrow (p \Rightarrow q) \in \mathrm{HP\_TAUT}$.

(28) $q \Rightarrow ((q \Rightarrow p) \Rightarrow p) \in \mathrm{HP\_TAUT}$.

(29) If $s \Rightarrow (q \Rightarrow p) \in \mathrm{HP\_TAUT}$ and $q \in \mathrm{HP\_TAUT}$, then $s \Rightarrow p \in \mathrm{HP\_TAUT}$.

3. Conjunctional Part of the Calculus

The following propositions are true:

(30) $p \Rightarrow p \wedge p \in \mathrm{HP\_TAUT}$.

(31) $p \wedge q \in$ HP_TAUT iff $p \in$ HP_TAUT and $q \in$ HP_TAUT .

(32) $p \wedge q \in$ HP_TAUT iff $q \wedge p \in$ HP_TAUT .

(33) $(p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r)) \in$ HP_TAUT .

(34) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow (p \wedge q \Rightarrow r) \in$ HP_TAUT .

(35) $(r \Rightarrow p) \Rightarrow ((r \Rightarrow q) \Rightarrow (r \Rightarrow p \wedge q)) \in$ HP_TAUT .

(36) $(p \Rightarrow q) \wedge p \Rightarrow q \in$ HP_TAUT .

(37) $(p \Rightarrow q) \wedge p \wedge s \Rightarrow q \in$ HP_TAUT .

(38) $(q \Rightarrow s) \Rightarrow (p \wedge q \Rightarrow s) \in$ HP_TAUT .

(39) $(q \Rightarrow s) \Rightarrow (q \wedge p \Rightarrow s) \in$ HP_TAUT .

(40) $(p \wedge s \Rightarrow q) \Rightarrow (p \wedge s \Rightarrow q \wedge s) \in$ HP_TAUT .

(41) $(p \Rightarrow q) \Rightarrow (p \wedge s \Rightarrow q \wedge s) \in$ HP_TAUT .

(42) $(p \Rightarrow q) \wedge (p \wedge s) \Rightarrow q \wedge s \in$ HP_TAUT .

(43) $p \wedge q \Rightarrow q \wedge p \in$ HP_TAUT .

(44) $(p \Rightarrow q) \wedge (p \wedge s) \Rightarrow s \wedge q \in$ HP_TAUT .

(45) $(p \Rightarrow q) \Rightarrow (p \wedge s \Rightarrow s \wedge q) \in$ HP_TAUT .

(46) $(p \Rightarrow q) \Rightarrow (s \wedge p \Rightarrow s \wedge q) \in$ HP_TAUT .

(47) $p \wedge (s \wedge q) \Rightarrow p \wedge (q \wedge s) \in$ HP_TAUT .

(48) $(p \Rightarrow q) \wedge (p \Rightarrow s) \Rightarrow (p \Rightarrow q \wedge s) \in$ HP_TAUT .

(49) $p \wedge q \wedge s \Rightarrow p \wedge (q \wedge s) \in$ HP_TAUT .

(50) $p \wedge (q \wedge s) \Rightarrow p \wedge q \wedge s \in$ HP_TAUT .

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