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The Definition of the Riemann Definite Integral and some Related Lemmas

Noboru Endou
Shinshu University
Nagano

Artur Korniłowicz[1]
University of Bialystok

Summary. This article introduces the Riemann definite integral on the closed interval of real. We present the definitions and related lemmas of the closed interval. We formalize the concept of the Riemann definite integral and the division of the closed interval of real, and prove the additivity of the integral.

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The notation and terminology used in this paper are introduced in the following papers: [28], [31], [8], [14], [2], [5], [6], [30], [22], [32], [18], [15], [7], [20], [26], [10], [12], [3], [27], [21], [4], [29], [16], [17], [24], [9], [11], [19], [25], [13], [23], and [1].

1. Definition of Closed Interval and its Properties

For simplicity, we adopt the following rules: a, a_1, a_2, b, b_1, b_2 are real numbers, p is a finite sequence, F, G, H are finite sequences of elements of $\mathbb{R}$, i, j, k are natural numbers, f is a function from $\mathbb{R}$ into $\mathbb{R}$, and x_1 is a set.

Let I_1 be a subset of $\mathbb{R}$. We say that I_1 is closed-interval if and only if:

(Def. 1) There exist real numbers a, b such that $a \leqslant b$ and $I_1 = [a, b]$.

Let us mention that there exists a subset of $\mathbb{R}$ which is closed-interval.

In the sequel A, A_1, A_2 are closed-interval subsets of $\mathbb{R}$.

The following propositions are true:

(1) Every closed-interval subset of $\mathbb{R}$ is compact.

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(2) If A is a closed-interval subset of $\mathbb{R}$, then A is non empty.

Let us observe that every subset of $\mathbb{R}$ which is closed-interval is also non empty and compact.

The following proposition is true

(3) If A is a closed-interval subset of $\mathbb{R}$, then A is lower bounded and upper bounded.

Let us observe that every subset of $\mathbb{R}$ which is closed-interval is also bounded.

One can verify that there exists a subset of $\mathbb{R}$ which is closed-interval.

Next we state three propositions:

(4) If A is a closed-interval subset of $\mathbb{R}$, then there exist a, b such that $a \leqslant b$ and $a = \inf A$ and $b = \sup A$.

(5) If A is a closed-interval subset of $\mathbb{R}$, then $A = [\inf A, \sup A]$.

(6) If $A = [a_1, b_1]$ and $A = [a_2, b_2]$, then $a_1 = a_2$ and $b_1 = b_2$.

2. Definition of Division of Closed Interval and its Properties

Let A be a closed-interval subset of $\mathbb{R}$. A non empty increasing finite sequence of elements of $\mathbb{R}$ is said to be a DivisionPoint of A if:

(Def. 2) $\operatorname{rng} \mathrm{it} \subseteq A$ and $\mathrm{it}(\operatorname{len} \mathrm{it}) = \sup A$.

Let A be a closed-interval subset of $\mathbb{R}$. The functor $\operatorname{divs} A$ yielding a set is defined by:

(Def. 3) $x_1 \in \operatorname{divs} A$ iff x_1 is a DivisionPoint of A.

Let A be a closed-interval subset of $\mathbb{R}$. One can check that $\operatorname{divs} A$ is non empty.

Let A be a closed-interval subset of $\mathbb{R}$. A non empty set is called a Division of A if:

(Def. 4) $x_1 \in \mathrm{it}$ iff x_1 is a DivisionPoint of A.

Let A be a closed-interval subset of $\mathbb{R}$. Observe that there exists a Division of A which is non empty.

The following proposition is true

(7) For every closed-interval subset A of $\mathbb{R}$ and for every non empty Division S of A holds every element of S is a DivisionPoint of A.

Let A be a closed-interval subset of $\mathbb{R}$ and let S be a non empty Division of A. We see that the element of S is a DivisionPoint of A.

In the sequel S denotes a non empty Division of A and D, D_1, D_2 denote elements of S.

Next we state two propositions:

(8) If $i \in \operatorname{dom} D$, then $D(i) \in A$.

(9) If $i \in \operatorname{dom} D$ and $i \neq 1$, then $i - 1 \in \operatorname{dom} D$ and $D(i-1) \in A$ and $i - 1 \in \mathbb{N}$.

Let A be a closed-interval subset of $\mathbb{R}$, let S be a non empty Division of A, let D be an element of S, and let i be a natural number. Let us assume that $i \in \operatorname{dom} D$. The functor $\operatorname{divset}(D, i)$ yielding a closed-interval subset of $\mathbb{R}$ is defined as follows:

(Def. 5)(i) $\inf \operatorname{divset}(D, i) = \inf A$ and $\sup \operatorname{divset}(D, i) = D(i)$ if $i = 1$,

(ii) $\inf \operatorname{divset}(D, i) = D(i-1)$ and $\sup \operatorname{divset}(D, i) = D(i)$, otherwise.

Next we state the proposition

(10) If $i \in \operatorname{dom} D$, then $\operatorname{divset}(D, i) \subseteq A$.

Let A be a subset of $\mathbb{R}$. The functor $\operatorname{vol}(A)$ yielding a real number is defined by:

(Def. 6) $\operatorname{vol}(A) = \sup A - \inf A$.

One can prove the following proposition

(11) For every closed-interval subset A of $\mathbb{R}$ holds $0 \leqslant \operatorname{vol}(A)$.

3. Definitions of Integrability and Related Topics

Let A be a closed-interval subset of $\mathbb{R}$, let f be a partial function from A to $\mathbb{R}$, let S be a non empty Division of A, and let D be an element of S. The functor $\text{upper\_volume}(f, D)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def. 7) $\operatorname{len} \text{upper\_volume}(f, D) = \operatorname{len} D$ and for every i such that $i \in \operatorname{Seg} \operatorname{len} D$ holds $(\text{upper\_volume}(f, D))(i) = \sup \operatorname{rng}(f \restriction \operatorname{divset}(D, i)) \cdot \operatorname{vol}(\operatorname{divset}(D, i))$.

The functor $\text{lower\_volume}(f, D)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined by:

(Def. 8) $\operatorname{len} \text{lower\_volume}(f, D) = \operatorname{len} D$ and for every i such that $i \in \operatorname{Seg} \operatorname{len} D$ holds $(\text{lower\_volume}(f, D))(i) = \inf \operatorname{rng}(f \restriction \operatorname{divset}(D, i)) \cdot \operatorname{vol}(\operatorname{divset}(D, i))$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a partial function from A to $\mathbb{R}$, let S be a non empty Division of A, and let D be an element of S. The functor $\text{upper\_sum}(f, D)$ yields a real number and is defined by:

(Def. 9) $\text{upper\_sum}(f, D) = \sum \text{upper\_volume}(f, D)$.

The functor $\text{lower\_sum}(f, D)$ yields a real number and is defined by:

(Def. 10) $\text{lower\_sum}(f, D) = \sum \text{lower\_volume}(f, D)$.

Let A be a closed-interval subset of $\mathbb{R}$. Then $\operatorname{divs} A$ is a Division of A.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from A to $\mathbb{R}$. The functor upper_sum_set f yielding a partial function from divs A to $\mathbb{R}$ is defined as follows:

(Def. 11) dom upper_sum_set f = divs A and for every element D of divs A such that $D \in$ dom upper_sum_set f holds (upper_sum_set f)(D) = upper_sum(f, D).

The functor lower_sum_set f yields a partial function from divs A to $\mathbb{R}$ and is defined as follows:

(Def. 12) dom lower_sum_set f = divs A and for every element D of divs A such that $D \in$ dom lower_sum_set f holds (lower_sum_set f)(D) = lower_sum(f, D).

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from A to $\mathbb{R}$. We say that f is upper integrable on A if and only if:

(Def. 13) rng upper_sum_set f is lower bounded.

We say that f is lower integrable on A if and only if:

(Def. 14) rng lower_sum_set f is upper bounded.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from A to $\mathbb{R}$. The functor upper_integral f yielding a real number is defined by:

(Def. 15) upper_integral f = inf rng upper_sum_set f.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from A to $\mathbb{R}$. The functor lower_integral f yields a real number and is defined as follows:

(Def. 16) lower_integral f = sup rng lower_sum_set f.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from A to $\mathbb{R}$. We say that f is integrable on A if and only if:

(Def. 17) f is upper integrable on A and f is lower integrable on A and upper_integral f = lower_integral f.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from A to $\mathbb{R}$. The functor integral f yields a real number and is defined by:

(Def. 18) integral f = upper_integral f.

4. Real Function's Properties

Next we state several propositions:

(12) For every non empty set X and for all partial functions f, g from X to $\mathbb{R}$ holds $\operatorname{rng}(f+g) \subseteq \operatorname{rng} f + \operatorname{rng} g$.

(13) Let A be a closed-interval subset of $\mathbb{R}$ and f be a partial function from A to $\mathbb{R}$. If f is lower bounded on A, then rng f is lower bounded.

(14) Let A be a closed-interval subset of $\mathbb{R}$ and f be a partial function from A to $\mathbb{R}$. If $\operatorname{rng} f$ is lower bounded, then f is lower bounded on A.

(15) Let A be a closed-interval subset of $\mathbb{R}$ and f be a partial function from A to $\mathbb{R}$. If f is upper bounded on A, then $\operatorname{rng} f$ is upper bounded.

(16) Let A be a closed-interval subset of $\mathbb{R}$ and f be a partial function from A to $\mathbb{R}$. If $\operatorname{rng} f$ is upper bounded, then f is upper bounded on A.

(17) Let A be a closed-interval subset of $\mathbb{R}$ and f be a partial function from A to $\mathbb{R}$. If f is bounded on A, then $\operatorname{rng} f$ is bounded.

5. Characteristic Function's Properties

The following propositions are true:

(18) For every closed-interval subset A of $\mathbb{R}$ holds $\chi_{A,A}$ is a constant on A.

(19) For every closed-interval subset A of $\mathbb{R}$ holds $\operatorname{rng}(\chi_{A,A}) = \{1\}$.

(20) For every closed-interval subset A of $\mathbb{R}$ and for every set B such that $B \cap \operatorname{dom}(\chi_{A,A}) \neq \emptyset$ holds $\operatorname{rng}(\chi_{A,A} \restriction B) = \{1\}$.

(21) If $i \in \operatorname{Seg} \operatorname{len} D$, then $\operatorname{vol}(\operatorname{divset}(D, i)) = (\text{lower\_volume}(\chi_{A,A}, D))(i)$.

(22) If $i \in \operatorname{Seg} \operatorname{len} D$, then $\operatorname{vol}(\operatorname{divset}(D, i)) = (\text{upper\_volume}(\chi_{A,A}, D))(i)$.

(23) If $\operatorname{len} F = \operatorname{len} G$ and $\operatorname{len} F = \operatorname{len} H$ and for every k such that $k \in \operatorname{dom} F$ holds $H(k) = F_k + G_k$, then $\sum H = \sum F + \sum G$.

(24) If $\operatorname{len} F = \operatorname{len} G$ and $\operatorname{len} F = \operatorname{len} H$ and for every k such that $k \in \operatorname{dom} F$ holds $H(k) = F_k - G_k$, then $\sum H = \sum F - \sum G$.

(25) Let A be a closed-interval subset of $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Then $\sum \text{lower\_volume}(\chi_{A,A}, D) = \operatorname{vol}(A)$.

(26) Let A be a closed-interval subset of $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Then $\sum \text{upper\_volume}(\chi_{A,A}, D) = \operatorname{vol}(A)$.

6. Some Properties of Darboux Sum

Let A be a closed-interval subset of $\mathbb{R}$, let f be a partial function from A to $\mathbb{R}$, let S be a non empty Division of A, and let D be an element of S. Then upper_volume(f, D) is a non empty finite sequence of elements of $\mathbb{R}$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a partial function from A to $\mathbb{R}$, let S be a non empty Division of A, and let D be an element of S. Then lower_volume(f, D) is a non empty finite sequence of elements of $\mathbb{R}$.

One can prove the following propositions:

(27) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is total and lower bounded on A, then $\inf \operatorname{rng} f \cdot \operatorname{vol}(A) \leqslant \text{lower\_sum}(f, D)$.

(28) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and i be a natural number. Suppose f is total and upper bounded on A and $i \in \operatorname{Seg} \operatorname{len} D$. Then $\sup \operatorname{rng} f \cdot \operatorname{vol}(\operatorname{divset}(D, i)) \geqslant \sup \operatorname{rng}(f \restriction \operatorname{divset}(D, i)) \cdot \operatorname{vol}(\operatorname{divset}(D, i))$.

(29) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is total and upper bounded on A, then $\text{upper\_sum}(f, D) \leqslant \sup \operatorname{rng} f \cdot \operatorname{vol}(A)$.

(30) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is total and bounded on A, then $\text{lower\_sum}(f, D) \leqslant \text{upper\_sum}(f, D)$.

Let x be a non empty finite sequence of elements of $\mathbb{R}$. Then $\operatorname{rng} x$ is a finite non empty subset of $\mathbb{R}$.

Let A be a closed-interval subset of $\mathbb{R}$ and let D be an element of $\operatorname{divs} A$. The functor δ_D yielding a real number is defined by:

(Def. 19) $\delta_D = \max \operatorname{rng} \text{upper\_volume}(\chi_{A,A}, D)$.

Let A be a closed-interval subset of $\mathbb{R}$, let S be a non empty Division of A, and let D_1, D_2 be elements of S. The predicate $D_1 \leqslant D_2$ is defined as follows:

(Def. 20) $\operatorname{len} D_1 \leqslant \operatorname{len} D_2$ and $\operatorname{rng} D_1 \subseteq \operatorname{rng} D_2$.

We introduce $D_2 \geqslant D_1$ as a synonym of $D_1 \leqslant D_2$.

One can prove the following propositions:

(31) Let A be a closed-interval subset of $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. If $\operatorname{len} D_1 = 1$, then $D_1 \leqslant D_2$.

(32) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. If f is total and upper bounded on A and $\operatorname{len} D_1 = 1$, then $\text{upper\_sum}(f, D_1) \geqslant \text{upper\_sum}(f, D_2)$.

(33) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. If f is total and lower bounded on A and $\operatorname{len} D_1 = 1$, then $\text{lower\_sum}(f, D_1) \leqslant \text{lower\_sum}(f, D_2)$.

(34) Let A be a closed-interval subset of $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If $i \in \operatorname{dom} D$, then there exist A_1, A_2 such that $A_1 = [\inf A, D(i)]$ and $A_2 = [D(i), \sup A]$ and $A = A_1 \cup A_2$.

(35) Let A be a closed-interval subset of $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. If $i \in \operatorname{dom} D_1$, then if $D_1 \leqslant D_2$, then there exists j such that $j \in \operatorname{dom} D_2$ and $D_1(i) = D_2(j)$.

Let A be a closed-interval subset of $\mathbb{R}$, let S be a non empty Division of A, let D_1, D_2 be elements of S, and let i be a natural number. Let us assume that $D_1 \leqslant D_2$. The functor $\mathrm{indx}(D_2, D_1, i)$ yields a natural number and is defined as follows:

(Def. 21)(i) $\mathrm{indx}(D_2, D_1, i) \in \mathrm{dom}\, D_2$ and $D_1(i) = D_2(\mathrm{indx}(D_2, D_1, i))$ if $i \in \mathrm{dom}\, D_1$,

(ii) $\mathrm{indx}(D_2, D_1, i) = 0$, otherwise.

Next we state four propositions:

(36) Let p be an increasing finite sequence of elements of $\mathbb{R}$ and n be a natural number. Suppose $n \leqslant \mathrm{len}\, p$. Then $p_{\restriction n}$ is an increasing finite sequence of elements of $\mathbb{R}$.

(37) Let p be an increasing finite sequence of elements of $\mathbb{R}$ and i, j be natural numbers. Suppose $j \in \mathrm{dom}\, p$ and $i \leqslant j$. Then $\mathrm{mid}(p, i, j)$ is an increasing finite sequence of elements of $\mathbb{R}$.

(38) Let A be a closed-interval subset of $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and i, j be natural numbers. Suppose $i \in \mathrm{dom}\, D$ and $j \in \mathrm{dom}\, D$ and $i \leqslant j$. Then there exists a closed-interval subset B of $\mathbb{R}$ such that $\inf B = (\mathrm{mid}(D, i, j))(1)$ and $\sup B = (\mathrm{mid}(D, i, j))(\mathrm{len}\, \mathrm{mid}(D, i, j))$ and $\mathrm{len}\, \mathrm{mid}(D, i, j) = (j - i) + 1$ and $\mathrm{mid}(D, i, j)$ is a DivisionPoint of B.

(39) Let A, B be closed-interval subsets of $\mathbb{R}$, S be a non empty Division of A, S_1 be a non empty Division of B, D be an element of S, and i, j be natural numbers. Suppose $i \in \mathrm{dom}\, D$ and $j \in \mathrm{dom}\, D$ and $i \leqslant j$ and $D(i) \geqslant \inf B$ and $D(j) = \sup B$. Then $\mathrm{mid}(D, i, j)$ is an element of S_1.

Let p be a finite sequence of elements of $\mathbb{R}$. The functor $\mathrm{PartSums}\, p$ yielding a finite sequence of elements of $\mathbb{R}$ is defined by:

(Def. 22) $\mathrm{len}\, \mathrm{PartSums}\, p = \mathrm{len}\, p$ and for every i such that $i \in \mathrm{Seg}\, \mathrm{len}\, p$ holds $(\mathrm{PartSums}\, p)(i) = \sum (p \restriction i)$.

We now state a number of propositions:

(40) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. Suppose $D_1 \leqslant D_2$ and f is total and upper bounded on A. Let i be a non empty natural number. If $i \in \mathrm{dom}\, D_1$, then $\sum (\mathrm{upper\_volume}(f, D_1) \restriction i) \geqslant \sum (\mathrm{upper\_volume}(f, D_2) \restriction \mathrm{indx}(D_2, D_1, i))$.

(41) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. Suppose $D_1 \leqslant D_2$ and f is total and lower bounded on A. Let i be a non empty natural number. If $i \in \mathrm{dom}\, D_1$, then $\sum (\mathrm{lower\_volume}(f, D_1) \restriction i) \leqslant \sum (\mathrm{lower\_volume}(f, D_2) \restriction \mathrm{indx}(D_2, D_1, i))$.

(42) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, D_1, D_2 be elements of S, and i

be a natural number. Suppose $D_1 \leqslant D_2$ and $i \in \operatorname{dom} D_1$ and f is total and upper bounded on A. Then (PartSums upper_volume$(f, D_1))(i) \geqslant$ (PartSums upper_volume$(f, D_2))(\operatorname{indx}(D_2, D_1, i))$.

(43) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, D_1, D_2 be elements of S, and i be a natural number. Suppose $D_1 \leqslant D_2$ and $i \in \operatorname{dom} D_1$ and f is total and lower bounded on A. Then (PartSums lower_volume$(f, D_1))(i) \leqslant$ (PartSums lower_volume$(f, D_2))(\operatorname{indx}(D_2, D_1, i))$.

(44) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Then (PartSums upper_volume$(f, D))(\operatorname{len} D) =$ upper_sum(f, D).

(45) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Then (PartSums lower_volume$(f, D))(\operatorname{len} D) =$ lower_sum(f, D).

(46) Let A be a closed-interval subset of $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. If $D_1 \leqslant D_2$, then $\operatorname{indx}(D_2, D_1, \operatorname{len} D_1) = \operatorname{len} D_2$.

(47) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. If $D_1 \leqslant D_2$ and f is total and upper bounded on A, then upper_sum$(f, D_2) \leqslant$ upper_sum(f, D_1).

(48) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. If $D_1 \leqslant D_2$ and f is total and lower bounded on A, then lower_sum$(f, D_2) \geqslant$ lower_sum(f, D_1).

(49) Let A be a closed-interval subset of $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. Then there exists an element D of S such that $D_1 \leqslant D$ and $D_2 \leqslant D$.

(50) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D_1, D_2 be elements of S. If f is total and bounded on A, then lower_sum$(f, D_1) \leqslant$ upper_sum(f, D_2).

7. Additivity of Integral

One can prove the following propositions:

(51) Let A be a closed-interval subset of $\mathbb{R}$ and f be a partial function from A to $\mathbb{R}$. Suppose f is upper integrable on A and f is lower integrable on A and f is total and bounded on A. Then upper_integral $f \geqslant$ lower_integral f.

(52) For all subsets X, Y of $\mathbb{R}$ holds $-X + -Y = -(X + Y)$.

(53) For all subsets X, Y of $\mathbb{R}$ such that X is upper bounded and $Y \neq \emptyset$ and Y is upper bounded holds $X + Y$ is upper bounded.

(54) For all non empty subsets X, Y of $\mathbb{R}$ such that X is upper bounded and Y is upper bounded holds $\sup(X + Y) = \sup X + \sup Y$.

(55) Let A be a closed-interval subset of $\mathbb{R}$, f, g be partial functions from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Suppose $i \in \operatorname{Seg} \operatorname{len} D$ and f is upper bounded on A and total and g is upper bounded on A and total. Then $(\text{upper\_volume}(f + g, D))(i) \leqslant (\text{upper\_volume}(f, D))(i) + (\text{upper\_volume}(g, D))(i)$.

(56) Let A be a closed-interval subset of $\mathbb{R}$, f, g be partial functions from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Suppose $i \in \operatorname{Seg} \operatorname{len} D$ and f is lower bounded on A and total and g is lower bounded on A and total. Then $(\text{lower\_volume}(f, D))(i) + (\text{lower\_volume}(g, D))(i) \leqslant (\text{lower\_volume}(f + g, D))(i)$.

(57) Let A be a closed-interval subset of $\mathbb{R}$, f, g be partial functions from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Suppose f is upper bounded on A and total and g is upper bounded on A and total. Then $\text{upper\_sum}(f + g, D) \leqslant \text{upper\_sum}(f, D) + \text{upper\_sum}(g, D)$.

(58) Let A be a closed-interval subset of $\mathbb{R}$, f, g be partial functions from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. Suppose f is lower bounded on A and total and g is lower bounded on A and total. Then $\text{lower\_sum}(f, D) + \text{lower\_sum}(g, D) \leqslant \text{lower\_sum}(f + g, D)$.

(59) Let X be a non empty set and f be a partial function from X to $\mathbb{R}$. If f is upper bounded on X and total, then $\operatorname{rng} f$ is upper bounded.

(60) Let X be a non empty set and f be a partial function from X to $\mathbb{R}$. If $\operatorname{rng} f$ is upper bounded and f is total, then f is upper bounded on X.

(61) Let X be a non empty set and f be a partial function from X to $\mathbb{R}$. If f is lower bounded on X and total, then $\operatorname{rng} f$ is lower bounded.

(62) Let X be a non empty set and f be a partial function from X to $\mathbb{R}$. If $\operatorname{rng} f$ is lower bounded and f is total, then f is lower bounded on X.

(63) Let A be a closed-interval subset of $\mathbb{R}$ and f, g be partial functions from A to $\mathbb{R}$. Suppose that
  (i) f is total and bounded on A,
  (ii) g is total and bounded on A,
  (iii) f is integrable on A, and
  (iv) g is integrable on A.

Then $f + g$ is integrable on A and $\operatorname{integral} f + g = \operatorname{integral} f + \operatorname{integral} g$.

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