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Bounded Domains and Unbounded Domains

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Summary. First, notions of inside components and outside components are introduced for any subset of n-dimensional Euclid space. Next, notions of the bounded domain and the unbounded domain are defined using the above components. If the dimension is larger than 1, and if a subset is bounded, a unbounded domain of the subset coincides with an outside component (which is unique) of the subset. For a sphere in n-dimensional space, the similar fact is true for a bounded domain. In 2 dimensional space, any rectangle also has such property. We discussed relations between the Jordan property and the concept of boundary, which are necessary to find points in domains near a curve. In the last part, we gave the sufficient criterion for belonging to the left component of some clockwise oriented finite sequences.

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The articles [44], [51], [12], [50], [53], [9], [10], [7], [22], [2], [1], [40], [54], [16], [27], [15], [24], [5], [38], [39], [20], [35], [32], [18], [42], [3], [8], [49], [46], [41], [21], [4], [26], [34], [37], [43], [6], [30], [52], [11], [25], [13], [17], [33], [14], [48], [47], [19], [23], [28], [29], [36], [45], and [31] provide the notation and terminology for this paper.

1. Definitions of Bounded Domain and Unbounded Domain

We follow the rules: m, n are natural numbers, r, s are real numbers, and x, y are sets.

The following propositions are true:

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(1) If $r \leqslant 0$, then $|r| = -r$.

(2) For all n, m such that $n \leqslant m$ and $m \leqslant n+2$ holds $m = n$ or $m = n+1$ or $m = n+2$.

(3) For all n, m such that $n \leqslant m$ and $m \leqslant n+3$ holds $m = n$ or $m = n+1$ or $m = n+2$ or $m = n+3$.

(4) For all n, m such that $n \leqslant m$ and $m \leqslant n+4$ holds $m = n$ or $m = n+1$ or $m = n+2$ or $m = n+3$ or $m = n+4$.

(5) For all real numbers a, b such that $a \geqslant 0$ and $b \geqslant 0$ holds $a+b \geqslant 0$.

(6) For all real numbers a, b such that $a > 0$ and $b \geqslant 0$ or $a \geqslant 0$ and $b > 0$ holds $a+b > 0$.

(7) For every finite sequence f such that $\mathrm{rng}\, f = \{x, y\}$ and $\mathrm{len}\, f = 2$ holds $f(1) = x$ and $f(2) = y$ or $f(1) = y$ and $f(2) = x$.

(8) Let f be an increasing finite sequence of elements of $\mathbb{R}$. If $\mathrm{rng}\, f = \{r, s\}$ and $\mathrm{len}\, f = 2$ and $r \leqslant s$, then $f(1) = r$ and $f(2) = s$.

In the sequel p, p_1, p_2, p_3, q, q_1, q_2 denote points of $\mathcal{E}^n_{\mathrm{T}}$.

We now state several propositions:

(9) $(p_1 + p_2) - p_3 = (p_1 - p_3) + p_2$.

(10) $||q|| = |q|$.

(11) $||q_1| - |q_2|| \leqslant |q_1 - q_2|$.

(12) $||[r]|| = |r|$.

(13) $q - 0_{\mathcal{E}^n_{\mathrm{T}}} = q$ and $0_{\mathcal{E}^n_{\mathrm{T}}} - q = -q$.

Let us consider n and let P be a subset of $\mathcal{E}^n_{\mathrm{T}}$. We say that P is n-convex if and only if:

(Def. 1) For all points w_1, w_2 of $\mathcal{E}^n_{\mathrm{T}}$ such that $w_1 \in P$ and $w_2 \in P$ holds $\mathcal{L}(w_1, w_2) \subseteq P$.

The following propositions are true:

(14) For every non empty subset P of $\mathcal{E}^n_{\mathrm{T}}$ such that P is n-convex holds P is connected.

(15) Let G be a non empty topological space, P be a subset of G, A be a subset of the carrier of G, and Q be a subset of $G \restriction A$. If $P \neq \emptyset$ and $P = Q$ and P is connected, then Q is connected.

Let us consider n and let A be a subset of $\mathcal{E}^n_{\mathrm{T}}$. We say that A is Bounded if and only if:

(Def. 2) There exists a subset C of the carrier of $\mathcal{E}^n$ such that $C = A$ and C is bounded.

One can prove the following proposition

(16) For all subsets A, B of $\mathcal{E}^n_{\mathrm{T}}$ such that B is Bounded and $A \subseteq B$ holds A is Bounded.

Let us consider n, let A be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, and let B be a subset of $\mathcal{E}^n_{\mathrm{T}}$. We say that B is inside component of A if and only if:

(Def. 3) B is a component of A^{c} and Bounded.

Next we state the proposition

(17) Let A be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and B be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Then B is inside component of A if and only if there exists a subset C of $(\mathcal{E}^n_{\mathrm{T}}) \restriction A^{\mathrm{c}}$ such that $C = B$ and C is a component of $(\mathcal{E}^n_{\mathrm{T}}) \restriction A^{\mathrm{c}}$ and for every subset D of the carrier of $\mathcal{E}^n$ such that $D = C$ holds D is bounded.

Let us consider n, let A be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, and let B be a subset of $\mathcal{E}^n_{\mathrm{T}}$. We say that B is outside component of A if and only if:

(Def. 4) B is a component of A^{c} and B is not Bounded.

Next we state three propositions:

(18) Let A be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and B be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Then B is outside component of A if and only if there exists a subset C of $(\mathcal{E}^n_{\mathrm{T}}) \restriction A^{\mathrm{c}}$ such that $C = B$ and C is a component of $(\mathcal{E}^n_{\mathrm{T}}) \restriction A^{\mathrm{c}}$ and it is not true that for every subset D of the carrier of $\mathcal{E}^n$ such that $D = C$ holds D is bounded.

(19) For all subsets A, B of $\mathcal{E}^n_{\mathrm{T}}$ such that B is inside component of A holds $B \subseteq A^{\mathrm{c}}$.

(20) For all subsets A, B of $\mathcal{E}^n_{\mathrm{T}}$ such that B is outside component of A holds $B \subseteq A^{\mathrm{c}}$.

Let us consider n and let A be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{BDD}\,A$ yields a subset of $\mathcal{E}^n_{\mathrm{T}}$ and is defined by:

(Def. 5) $\mathrm{BDD}\,A = \bigcup\{B; B$ ranges over subsets of $\mathcal{E}^n_{\mathrm{T}}$: B is inside component of $A\}$.

Let us consider n and let A be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{UBD}\,A$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by:

(Def. 6) $\mathrm{UBD}\,A = \bigcup\{B; B$ ranges over subsets of $\mathcal{E}^n_{\mathrm{T}}$: B is outside component of $A\}$.

One can prove the following propositions:

(21) $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ is n-convex.

(22) $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ is connected.

Let us consider n. One can check that $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ is connected.

We now state several propositions:

(23) $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ is a component of $\mathcal{E}^n_{\mathrm{T}}$.

(24) For every subset A of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{BDD}\,A$ is a union of components of $(\mathcal{E}^n_{\mathrm{T}}) \restriction A^{\mathrm{c}}$.

(25) For every subset A of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{UBD}\,A$ is a union of components of $(\mathcal{E}^n_{\mathrm{T}}) \restriction A^{\mathrm{c}}$.

(26) Let A be a subset of the carrier of $\mathcal{E}^n_{\rm T}$ and B be a subset of $\mathcal{E}^n_{\rm T}$. If B is inside component of A, then $B \subseteq \mathrm{BDD}\,A$.

(27) Let A be a subset of the carrier of $\mathcal{E}^n_{\rm T}$ and B be a subset of $\mathcal{E}^n_{\rm T}$. If B is outside component of A, then $B \subseteq \mathrm{UBD}\,A$.

(28) For every subset A of the carrier of $\mathcal{E}^n_{\rm T}$ holds $\mathrm{BDD}\,A \cap \mathrm{UBD}\,A = \emptyset$.

(29) For every subset A of the carrier of $\mathcal{E}^n_{\rm T}$ holds $\mathrm{BDD}\,A \subseteq A^{\rm c}$.

(30) For every subset A of the carrier of $\mathcal{E}^n_{\rm T}$ holds $\mathrm{UBD}\,A \subseteq A^{\rm c}$.

(31) For every subset A of the carrier of $\mathcal{E}^n_{\rm T}$ holds $\mathrm{BDD}\,A \cup \mathrm{UBD}\,A = A^{\rm c}$.

In the sequel u is a point of $\mathcal{E}^n$.

One can prove the following propositions:

(32) Let G be a non empty topological space, w_1, w_2, w_3 be points of G, h_1 be a map from $\mathbb{I}$ into G, and h_2 be a map from $\mathbb{I}$ into G. Suppose h_1 is continuous and $w_1 = h_1(0)$ and $w_2 = h_1(1)$ and h_2 is continuous and $w_2 = h_2(0)$ and $w_3 = h_2(1)$. Then there exists a map h_3 from $\mathbb{I}$ into G such that h_3 is continuous and $w_1 = h_3(0)$ and $w_3 = h_3(1)$ and $\mathrm{rng}\,h_3 \subseteq \mathrm{rng}\,h_1 \cup \mathrm{rng}\,h_2$.

(33) For every subset P of $\mathcal{E}^n_{\rm T}$ such that $P = \mathcal{R}^n$ holds P is connected.

Let us consider n. The functor $1 * n$ yielding a finite sequence of elements of $\mathbb{R}$ is defined by:

(Def. 7) $1 * n = n \mapsto (1$ **qua** real number$)$.

Let us consider n. Then $1 * n$ is an element of $\mathcal{R}^n$.

Let us consider n. The functor $1.\mathrm{REAL}\,n$ yielding a point of $\mathcal{E}^n_{\rm T}$ is defined by:

(Def. 8) $1.\mathrm{REAL}\,n = 1 * n$.

One can prove the following propositions:

(34) $|1 * n| = n \mapsto (1$ **qua** real number$)$.

(35) $|1 * n| = \sqrt{n}$.

(36) $1.\mathrm{REAL}\,1 = \langle (1$ **qua** real number$) \rangle$.

(37) $|\,1.\mathrm{REAL}\,n| = \sqrt{n}$.

(38) If $1 \leqslant n$, then $1 \leqslant |\,1.\mathrm{REAL}\,n|$.

(39) For every subset W of the carrier of $\mathcal{E}^n$ such that $n \geqslant 1$ and $W = \mathcal{R}^n$ holds W is not bounded.

(40) Let A be a subset of $\mathcal{E}^n_{\rm T}$. Then A is Bounded if and only if there exists a real number r such that for every point q of $\mathcal{E}^n_{\rm T}$ such that $q \in A$ holds $|q| < r$.

(41) If $n \geqslant 1$, then $\Omega_{\mathcal{E}^n_{\rm T}}$ is not Bounded.

(42) If $n \geqslant 1$, then $\mathrm{UBD}\,\emptyset_{\mathcal{E}^n_{\rm T}} = \mathcal{R}^n$.

(43) Let w_1, w_2, w_3 be points of $\mathcal{E}^n_{\rm T}$, P be a non empty subset of the carrier of $\mathcal{E}^n_{\rm T}$, and h_1, h_2 be maps from $\mathbb{I}$ into $(\mathcal{E}^n_{\rm T})\restriction P$. Suppose h_1 is continuous and $w_1 = h_1(0)$ and $w_2 = h_1(1)$ and h_2 is continuous and $w_2 = h_2(0)$ and $w_3 = h_2(1)$. Then there exists a map h_3 from $\mathbb{I}$ into $(\mathcal{E}^n_{\rm T})\restriction P$ such that h_3 is continuous and $w_1 = h_3(0)$ and $w_3 = h_3(1)$.

(44) Let P be a subset of the carrier of $\mathcal{E}^n_{\rm T}$ and w_1, w_2, w_3 be points of $\mathcal{E}^n_{\rm T}$. Suppose $w_1 \in P$ and $w_2 \in P$ and $w_3 \in P$ and $\mathcal{L}(w_1, w_2) \subseteq P$ and $\mathcal{L}(w_2, w_3) \subseteq P$. Then there exists a map h from $\mathbb{I}$ into $(\mathcal{E}^n_{\rm T})\restriction P$ such that h is continuous and $w_1 = h(0)$ and $w_3 = h(1)$.

(45) Let P be a subset of the carrier of $\mathcal{E}^n_{\rm T}$ and w_1, w_2, w_3, w_4 be points of $\mathcal{E}^n_{\rm T}$. Suppose $w_1 \in P$ and $w_2 \in P$ and $w_3 \in P$ and $w_4 \in P$ and $\mathcal{L}(w_1, w_2) \subseteq P$ and $\mathcal{L}(w_2, w_3) \subseteq P$ and $\mathcal{L}(w_3, w_4) \subseteq P$. Then there exists a map h from $\mathbb{I}$ into $(\mathcal{E}^n_{\rm T})\restriction P$ such that h is continuous and $w_1 = h(0)$ and $w_4 = h(1)$.

(46) Let P be a subset of the carrier of $\mathcal{E}^n_{\rm T}$ and w_1, w_2, w_3, w_4, w_5, w_6, w_7 be points of $\mathcal{E}^n_{\rm T}$. Suppose $w_1 \in P$ and $w_2 \in P$ and $w_3 \in P$ and $w_4 \in P$ and $w_5 \in P$ and $w_6 \in P$ and $w_7 \in P$ and $\mathcal{L}(w_1, w_2) \subseteq P$ and $\mathcal{L}(w_2, w_3) \subseteq P$ and $\mathcal{L}(w_3, w_4) \subseteq P$ and $\mathcal{L}(w_4, w_5) \subseteq P$ and $\mathcal{L}(w_5, w_6) \subseteq P$ and $\mathcal{L}(w_6, w_7) \subseteq P$. Then there exists a map h from $\mathbb{I}$ into $(\mathcal{E}^n_{\rm T})\restriction P$ such that h is continuous and $w_1 = h(0)$ and $w_7 = h(1)$.

(47) For all points w_1, w_2 of $\mathcal{E}^n_{\rm T}$ such that it is not true that there exists a real number r such that $w_1 = r \cdot w_2$ or $w_2 = r \cdot w_1$ holds $0_{\mathcal{E}^n_{\rm T}} \notin \mathcal{L}(w_1, w_2)$.

(48) Let w_1, w_2 be points of $\mathcal{E}^n_{\rm T}$ and P be a subset of $(\mathcal{E}^n)_{\rm top}$. Suppose $P = \mathcal{L}(w_1, w_2)$ and $0_{\mathcal{E}^n_{\rm T}} \notin \mathcal{L}(w_1, w_2)$. Then there exists a point w_0 of $\mathcal{E}^n_{\rm T}$ such that $w_0 \in \mathcal{L}(w_1, w_2)$ and $|w_0| > 0$ and $|w_0| = ({\rm dist}_{\rm min}(P))(0_{\mathcal{E}^n_{\rm T}})$.

(49) Let a be a real number, Q be a subset of the carrier of $\mathcal{E}^n_{\rm T}$, and w_1, w_4 be points of $\mathcal{E}^n_{\rm T}$. Suppose $Q = \{q : |q| > a\}$ and $w_1 \in Q$ and $w_4 \in Q$ and it is not true that there exists a real number r such that $w_1 = r \cdot w_4$ or $w_4 = r \cdot w_1$. Then there exist points w_2, w_3 of $\mathcal{E}^n_{\rm T}$ such that $w_2 \in Q$ and $w_3 \in Q$ and $\mathcal{L}(w_1, w_2) \subseteq Q$ and $\mathcal{L}(w_2, w_3) \subseteq Q$ and $\mathcal{L}(w_3, w_4) \subseteq Q$.

(50) Let a be a real number, Q be a subset of the carrier of $\mathcal{E}^n_{\rm T}$, and w_1, w_4 be points of $\mathcal{E}^n_{\rm T}$. Suppose $Q = \mathcal{R}^n \setminus \{q : |q| < a\}$ and $w_1 \in Q$ and $w_4 \in Q$ and it is not true that there exists a real number r such that $w_1 = r \cdot w_4$ or $w_4 = r \cdot w_1$. Then there exist points w_2, w_3 of $\mathcal{E}^n_{\rm T}$ such that $w_2 \in Q$ and $w_3 \in Q$ and $\mathcal{L}(w_1, w_2) \subseteq Q$ and $\mathcal{L}(w_2, w_3) \subseteq Q$ and $\mathcal{L}(w_3, w_4) \subseteq Q$.

(51) Let x be an element of $\mathcal{R}^n$. Then x is a finite sequence of elements of $\mathbb{R}$ and for every finite sequence f such that $f = x$ holds ${\rm len}\, f = n$.

(52) Every finite sequence f of elements of $\mathbb{R}$ is an element of $\mathcal{R}^{{\rm len}\, f}$ and a point of $\mathcal{E}^{{\rm len}\, f}_{\rm T}$.

(53) Let x be an element of $\mathcal{R}^n$, f, g be finite sequences of elements of $\mathbb{R}$, and r be a real number. Suppose $f = x$ and $g = r \cdot x$. Then ${\rm len}\, f = {\rm len}\, g$ and for

every natural number i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} f$ holds $\pi_i g = r \cdot \pi_i f$.

(54) Let x be an element of $\mathcal{R}^n$ and f be a finite sequence. Suppose $x \neq \langle \underbrace{0, \ldots, 0}_{n} \rangle$ and $x = f$. Then there exists a natural number i such that $1 \leqslant i$ and $i \leqslant n$ and $f(i) \neq 0$.

(55) Let x be an element of $\mathcal{R}^n$. Suppose $n \geqslant 2$ and $x \neq \langle \underbrace{0, \ldots, 0}_{n} \rangle$. Then it is not true that there exists an element y of $\mathcal{R}^n$ and there exists a real number r such that $y = r \cdot x$ or $x = r \cdot y$.

(56) Let a be a real number, Q be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, and w_1, w_7 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $n \geqslant 2$ and $Q = \{q : |q| > a\}$ and $w_1 \in Q$ and $w_7 \in Q$ and there exists a real number r such that $w_1 = r \cdot w_7$ or $w_7 = r \cdot w_1$. Then there exist points w_2, w_3, w_4, w_5, w_6 of $\mathcal{E}^n_{\mathrm{T}}$ such that $w_2 \in Q$ and $w_3 \in Q$ and $w_4 \in Q$ and $w_5 \in Q$ and $w_6 \in Q$ and $\mathcal{L}(w_1, w_2) \subseteq Q$ and $\mathcal{L}(w_2, w_3) \subseteq Q$ and $\mathcal{L}(w_3, w_4) \subseteq Q$ and $\mathcal{L}(w_4, w_5) \subseteq Q$ and $\mathcal{L}(w_5, w_6) \subseteq Q$ and $\mathcal{L}(w_6, w_7) \subseteq Q$.

(57) Let a be a real number, Q be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, and w_1, w_7 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $n \geqslant 2$ and $Q = \mathcal{R}^n \setminus \{q : |q| < a\}$ and $w_1 \in Q$ and $w_7 \in Q$ and there exists a real number r such that $w_1 = r \cdot w_7$ or $w_7 = r \cdot w_1$. Then there exist points w_2, w_3, w_4, w_5, w_6 of $\mathcal{E}^n_{\mathrm{T}}$ such that $w_2 \in Q$ and $w_3 \in Q$ and $w_4 \in Q$ and $w_5 \in Q$ and $w_6 \in Q$ and $\mathcal{L}(w_1, w_2) \subseteq Q$ and $\mathcal{L}(w_2, w_3) \subseteq Q$ and $\mathcal{L}(w_3, w_4) \subseteq Q$ and $\mathcal{L}(w_4, w_5) \subseteq Q$ and $\mathcal{L}(w_5, w_6) \subseteq Q$ and $\mathcal{L}(w_6, w_7) \subseteq Q$.

(58) For every real number a such that $n \geqslant 1$ holds $\{q : |q| > a\} \neq \emptyset$.

(59) For every real number a and for every subset P of $\mathcal{E}^n_{\mathrm{T}}$ such that $n \geqslant 2$ and $P = \{q : |q| > a\}$ holds P is connected.

(60) For every real number a such that $n \geqslant 1$ holds $\mathcal{R}^n \setminus \{q : |q| < a\} \neq \emptyset$.

(61) For every real number a and for every subset P of $\mathcal{E}^n_{\mathrm{T}}$ such that $n \geqslant 2$ and $P = \mathcal{R}^n \setminus \{q : |q| < a\}$ holds P is connected.

(62) Let a be a real number, n be a natural number, and P be a subset of $\mathcal{E}^n_{\mathrm{T}}$. If $n \geqslant 1$ and $P = \mathcal{R}^n \setminus \{q; q$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $|q| < a\}$, then P is not Bounded.

(63) Let a be a real number and P be a subset of $\mathcal{E}^1_{\mathrm{T}}$. Suppose $P = \{q; q$ ranges over points of $\mathcal{E}^1_{\mathrm{T}}$: $\bigvee_r (q = \langle r \rangle \ \wedge \ r > a)\}$. Then P is n-convex.

(64) Let a be a real number and P be a subset of $\mathcal{E}^1_{\mathrm{T}}$. Suppose $P = \{q; q$ ranges over points of $\mathcal{E}^1_{\mathrm{T}}$: $\bigvee_r (q = \langle r \rangle \ \wedge \ r < -a)\}$. Then P is n-convex.

(65) Let a be a real number and P be a subset of $\mathcal{E}^1_{\mathrm{T}}$. Suppose $P = \{q; q$ ranges over points of $\mathcal{E}^1_{\mathrm{T}}$: $\bigvee_r (q = \langle r \rangle \ \wedge \ r > a)\}$. Then P is connected.

(66) Let a be a real number and P be a subset of $\mathcal{E}^1_{\mathrm{T}}$. Suppose $P = \{q; q$ ranges over points of $\mathcal{E}^1_{\mathrm{T}}$: $\bigvee_r (q = \langle r \rangle \ \wedge \ r < -a)\}$. Then P is connected.

(67) Let W be a subset of the carrier of $\mathcal{E}^1$, a be a real number, and P be a subset of $\mathcal{E}^1_{\mathrm{T}}$. Suppose $W = \{q; q$ ranges over points of $\mathcal{E}^1_{\mathrm{T}}$: $\bigvee_r$ $(q = \langle r \rangle \ \wedge \ r > a)\}$ and $P = W$. Then P is connected and W is not bounded.

(68) Let W be a subset of the carrier of $\mathcal{E}^1$, a be a real number, and P be a subset of $\mathcal{E}^1_{\mathrm{T}}$. Suppose $W = \{q; q$ ranges over points of $\mathcal{E}^1_{\mathrm{T}}$: $\bigvee_r$ $(q = \langle r \rangle \ \wedge \ r < -a)\}$ and $P = W$. Then P is connected and W is not bounded.

(69) Let W be a subset of the carrier of $\mathcal{E}^n$, a be a real number, and P be a subset of $\mathcal{E}^n_{\mathrm{T}}$. If $n \geqslant 2$ and $W = \{q : |q| > a\}$ and $P = W$, then P is connected and W is not bounded.

(70) Let W be a subset of the carrier of $\mathcal{E}^n$, a be a real number, and P be a subset of $\mathcal{E}^n_{\mathrm{T}}$. If $n \geqslant 2$ and $W = \mathcal{R}^n \setminus \{q : |q| < a\}$ and $P = W$, then P is connected and W is not bounded.

(71) Let P, P_1 be subsets of $\mathcal{E}^n_{\mathrm{T}}$, Q be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, and W be a subset of the carrier of $\mathcal{E}^n$. Suppose $P = W$ and P is connected and W is not bounded and $P_1 = \mathrm{Component}(\mathrm{Down}(P, Q^{\mathrm{c}}))$ and $W \cap Q = \emptyset$. Then P_1 is outside component of Q.

Let S be a 1-sorted structure and let A be a subset of the carrier of S. The functor $\mathrm{RAC}\,A$ yields a subset of S and is defined as follows:

(Def. 9) $\mathrm{RAC}\,A = A$.

The following propositions are true:

(72) Let A be a subset of the carrier of $\mathcal{E}^n$, B be a non empty subset of the carrier of $\mathcal{E}^n$, and C be a subset of the carrier of $\mathcal{E}^n \restriction B$. If $A \subseteq B$ and $A = C$ and C is bounded, then A is bounded.

(73) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ such that A is compact holds A is Bounded.

(74) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ such that $1 \leqslant n$ and A is Bounded holds $A^{\mathrm{c}} \neq \emptyset$.

(75) Let r be a real number. Then

(i) there exists a subset B of the carrier of $\mathcal{E}^n$ such that $B = \{q : |q| < r\}$, and

(ii) for every subset A of the carrier of $\mathcal{E}^n$ such that $A = \{q_1 : |q_1| < r\}$ holds A is bounded.

(76) Let A be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $n \geqslant 2$ and A is Bounded. Then there exists a subset B of $\mathcal{E}^n_{\mathrm{T}}$ such that B is outside component of A and $B = \mathrm{UBD}\,A$.

(77) For every real number a and for every subset P of $\mathcal{E}^n_{\mathrm{T}}$ such that $P = \{q : |q| < a\}$ holds P is n-convex.

(78) For every real number a and for every subset P of $\mathcal{E}^n_{\mathrm{T}}$ such that $P = \mathrm{Ball}(u, a)$ holds P is n-convex.

(79) For every real number a and for every subset P of $\mathcal{E}^n_{\mathrm{T}}$ such that $a > 0$ and $P = \{q : |q| < a\}$ holds P is connected.

In the sequel R denotes a subset of $\mathcal{E}^n_{\rm T}$, P denotes a subset of the carrier of $\mathcal{E}^n_{\rm T}$, and f denotes a finite sequence of elements of $\mathcal{E}^n_{\rm T}$.

Next we state a number of propositions:

(80) Suppose $p \neq q$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$. Then there exists a map h from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$ such that h is continuous and $h(0) = p$ and $h(1) = q$ and $\mathrm{rng}\, h \subseteq \mathrm{Ball}(u,r)$.

(81) Let f be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$. Suppose f is continuous and $f(0) = p_1$ and $f(1) = p_2$ and $p \in \mathrm{Ball}(u,r)$ and $p_2 \in \mathrm{Ball}(u,r)$. Then there exists a map h from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$ such that h is continuous and $h(0) = p_1$ and $h(1) = p$ and $\mathrm{rng}\, h \subseteq \mathrm{rng}\, f \cup \mathrm{Ball}(u,r)$.

(82) Let f be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$. Suppose $p \neq p_1$ and f is continuous and $\mathrm{rng}\, f \subseteq P$ and $f(0) = p_1$ and $f(1) = p_2$ and $p \in \mathrm{Ball}(u,r)$ and $p_2 \in \mathrm{Ball}(u,r)$ and $\mathrm{Ball}(u,r) \subseteq P$. Then there exists a map f_1 from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$ such that f_1 is continuous and $\mathrm{rng}\, f_1 \subseteq P$ and $f_1(0) = p_1$ and $f_1(1) = p$.

(83) Let given p and P be a subset of $\mathcal{E}^n_{\rm T}$. Suppose that
- (i) R is connected and open, and
- (ii) $P = \{q : q \neq p \ \wedge \ q \in R \ \wedge \ \neg \bigvee_{f\,:\,\text{map from } \mathbb{I} \text{ into } \mathcal{E}^n_{\rm T}} (f \text{ is continuous} \wedge \ \mathrm{rng}\, f \subseteq R \ \wedge \ f(0) = p \ \wedge \ f(1) = q)\}$.

Then P is open.

(84) Let P be a subset of $\mathcal{E}^n_{\rm T}$. Suppose that
- (i) R is connected and open,
- (ii) $p \in R$, and
- (iii) $P = \{q : q = p \ \vee \ \bigvee_{f\,:\,\text{map from } \mathbb{I} \text{ into } \mathcal{E}^n_{\rm T}} (f \text{ is continuous} \wedge \ \mathrm{rng}\, f \subseteq R \ \wedge \ f(0) = p \ \wedge \ f(1) = q)\}$.

Then P is open.

(85) Let R be a subset of the carrier of $\mathcal{E}^n_{\rm T}$. Suppose $p \in R$ and $P = \{q : q = p \ \vee \ \bigvee_{f\,:\,\text{map from } \mathbb{I} \text{ into } \mathcal{E}^n_{\rm T}} (f \text{ is continuous} \wedge \ \mathrm{rng}\, f \subseteq R \ \wedge \ f(0) = p \ \wedge \ f(1) = q)\}$. Then $P \subseteq R$.

(86) Let R be a subset of $\mathcal{E}^n_{\rm T}$ and p be a point of $\mathcal{E}^n_{\rm T}$. Suppose that
- (i) R is connected and open,
- (ii) $p \in R$, and
- (iii) $P = \{q : q = p \ \vee \ \bigvee_{f\,:\,\text{map from } \mathbb{I} \text{ into } \mathcal{E}^n_{\rm T}} (f \text{ is continuous} \wedge \ \mathrm{rng}\, f \subseteq R \ \wedge \ f(0) = p \ \wedge \ f(1) = q)\}$.

Then $R \subseteq P$.

(87) Let R be a subset of $\mathcal{E}^n_{\rm T}$ and p, q be points of $\mathcal{E}^n_{\rm T}$. Suppose R is connected and open and $p \in R$ and $q \in R$ and $p \neq q$. Then there exists a map f from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$ such that f is continuous and $\mathrm{rng}\, f \subseteq R$ and $f(0) = p$ and $f(1) = q$.

(88) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ and for every real number a such that $A = \{q : |q| = a\}$ holds $-A$ is open and A is closed.

(89) For every non empty subset B of $\mathcal{E}^n_{\mathrm{T}}$ such that B is open holds $(\mathcal{E}^n_{\mathrm{T}}) \restriction B$ is locally connected.

(90) Let B be a non empty subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, A be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, and a be a real number. If $A = \{q : |q| = a\}$ and $A^{\mathrm{c}} = B$, then $(\mathcal{E}^n_{\mathrm{T}}) \restriction B$ is locally connected.

(91) For every map f from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ such that for every q holds $f(q) = |q|$ holds f is continuous.

(92) There exists a map f from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ such that for every q holds $f(q) = |q|$ and f is continuous.

Let X, Y be non empty 1-sorted structures, let f be a map from X into Y, and let x be a set. Let us assume that x is a point of X. The functor $\pi_x f$ yielding a point of Y is defined as follows:

(Def. 10) $\pi_x f = f(x)$.

We now state four propositions:

(93) Let g be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$. Suppose g is continuous. Then there exists a map f from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ such that for every point t of $\mathbb{I}$ holds $f(t) = |g(t)|$ and f is continuous.

(94) Let g be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$ and a be a real number. Suppose g is continuous and $|\pi_0 g| \leqslant a$ and $a \leqslant |\pi_1 g|$. Then there exists a point s of $\mathbb{I}$ such that $|\pi_s g| = a$.

(95) If $q = \langle r \rangle$, then $|q| = |r|$.

(96) Let A be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and a be a real number. Suppose $n \geqslant 1$ and $a > 0$ and $A = \{q : |q| = a\}$. Then there exists a subset B of $\mathcal{E}^n_{\mathrm{T}}$ such that B is inside component of A and $B = \mathrm{BDD}\, A$.

2. Bounded and Unbounded Domains of Rectangles

In the sequel D is a non vertical non horizontal non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$.

Next we state several propositions:

(97) len the Go-board of SpStSeq D = 2 and width the Go-board of SpStSeq D = 2 and π_1 SpStSeq D = (the Go-board of SpStSeq $D)_{1,2}$ and π_2 SpStSeq D = (the Go-board of SpStSeq $D)_{2,2}$ and π_3 SpStSeq D = (the Go-board of SpStSeq $D)_{2,1}$ and π_4 SpStSeq D = (the Go-board of SpStSeq $D)_{1,1}$ and π_5 SpStSeq D = (the Go-board of SpStSeq $D)_{1,2}$.

(98) LeftComp(SpStSeq D) is not Bounded.

(99) LeftComp(SpStSeq D) $\subseteq$ UBD $\widetilde{\mathcal{L}}$(SpStSeq D).

(100) Let G be a topological space and A, B, C be subsets of G. Suppose A is a component of G and B is a component of G and C is connected and $A \cap C \neq \emptyset$ and $B \cap C \neq \emptyset$. Then $A = B$.

(101) For every subset B of $\mathcal{E}^2_{\rm T}$ such that B is a component of $(\widetilde{\mathcal{L}}(\text{SpStSeq}\,D))^{\rm c}$ and B is not Bounded holds $B =$ LeftComp(SpStSeq D).

(102) RightComp(SpStSeq D) $\subseteq$ BDD $\widetilde{\mathcal{L}}$(SpStSeq D) and RightComp(SpStSeq D) is Bounded.

(103) LeftComp(SpStSeq D) $=$ UBD $\widetilde{\mathcal{L}}$(SpStSeq D) and RightComp(SpStSeq D) $=$ BDD $\widetilde{\mathcal{L}}$(SpStSeq D).

(104) UBD $\widetilde{\mathcal{L}}$(SpStSeq D) $\neq \emptyset$ and UBD $\widetilde{\mathcal{L}}$(SpStSeq D) is outside component of $\widetilde{\mathcal{L}}$(SpStSeq D) and BDD $\widetilde{\mathcal{L}}$(SpStSeq D) $\neq \emptyset$ and BDD $\widetilde{\mathcal{L}}$(SpStSeq D) is inside component of $\widetilde{\mathcal{L}}$(SpStSeq D).

3. Jordan Property and Boundary Property

One can prove the following propositions:

(105) Let G be a non empty topological space and A be a subset of G. Suppose $A^{\rm c} \neq \emptyset$. Then A is boundary if and only if for every set x and for every subset V of G such that $x \in A$ and $x \in V$ and V is open there exists a subset B of the carrier of G such that B is a component of $A^{\rm c}$ and $V \cap B \neq \emptyset$.

(106) Let A be a subset of $\mathcal{E}^2_{\rm T}$. Suppose $A^{\rm c} \neq \emptyset$. Then A is boundary and Jordan if and only if there exist subsets A_1, A_2 of $\mathcal{E}^2_{\rm T}$ such that $A^{\rm c} = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$ and $\overline{A_1} \setminus A_1 = \overline{A_2} \setminus A_2$ and $A = \overline{A_1} \setminus A_1$ and for all subsets C_1, C_2 of $(\mathcal{E}^2_{\rm T}) \restriction A^{\rm c}$ such that $C_1 = A_1$ and $C_2 = A_2$ holds C_1 is a component of $(\mathcal{E}^2_{\rm T}) \restriction A^{\rm c}$ and C_2 is a component of $(\mathcal{E}^2_{\rm T}) \restriction A^{\rm c}$.

(107) For every point p of $\mathcal{E}^n_{\rm T}$ and for every subset P of $\mathcal{E}^n_{\rm T}$ such that $n \geqslant 1$ and $P = \{p\}$ holds P is boundary.

(108) For all points p, q of $\mathcal{E}^2_{\rm T}$ and for every r such that $p_{\mathbf{1}} = q_{\mathbf{2}}$ and $-p_{\mathbf{2}} = q_{\mathbf{1}}$ and $p = r \cdot q$ holds $p_{\mathbf{1}} = 0$ and $p_{\mathbf{2}} = 0$ and $p = 0_{\mathcal{E}^2_{\rm T}}$.

(109) For all points q_1, q_2 of $\mathcal{E}^2_{\rm T}$ holds $\mathcal{L}(q_1, q_2)$ is boundary.

Let q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Observe that $\mathcal{L}(q_1, q_2)$ is boundary.

One can prove the following proposition

(110) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds $\widetilde{\mathcal{L}}(f)$ is boundary.

Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Note that $\widetilde{\mathcal{L}}(f)$ is boundary.

We now state several propositions:

(111) For every point e_1 of $\mathcal{E}^n$ and for all points p, q of $\mathcal{E}^n_{\rm T}$ such that $p = e_1$ and $q \in \mathrm{Ball}(e_1, r)$ holds $|p - q| < r$ and $|q - p| < r$.

(112) Let a be a real number and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $a > 0$ and $p \in \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$. Then there exists a point q of $\mathcal{E}^2_{\rm T}$ such that $q \in \mathrm{UBD}\, \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$ and $|p - q| < a$.

(113) $\mathcal{R}^0 = \{0_{\mathcal{E}^0_{\rm T}}\}$.

(114) For every subset A of $\mathcal{E}^n_{\rm T}$ such that A is Bounded holds $\mathrm{BDD}\, A$ is Bounded.

(115) Let G be a non empty topological space and A, B, C, D be subsets of G. Suppose A is a component of G and B is a component of G and C is a component of G and $A \cup B =$ the carrier of G and $C \cap A = \emptyset$. Then $C = B$.

(116) For every subset A of $\mathcal{E}^2_{\rm T}$ such that A is Bounded and Jordan holds $\mathrm{BDD}\, A$ is inside component of A.

(117) Let a be a real number and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $a > 0$ and $p \in \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$. Then there exists a point q of $\mathcal{E}^2_{\rm T}$ such that $q \in \mathrm{BDD}\, \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$ and $|p - q| < a$.

4. Points in LeftComp

In the sequel f denotes a clockwise oriented non constant standard special circular sequence.

Next we state four propositions:

(118) For every point p of $\mathcal{E}^2_{\rm T}$ such that $\pi_1 f = \text{N-min}\, \widetilde{\mathcal{L}}(f)$ and $p_{\mathbf{1}} < \text{W-bound}\, \widetilde{\mathcal{L}}(f)$ holds $p \in \mathrm{LeftComp}(f)$.

(119) For every point p of $\mathcal{E}^2_{\rm T}$ such that $\pi_1 f = \text{N-min}\, \widetilde{\mathcal{L}}(f)$ and $p_{\mathbf{1}} > \text{E-bound}\, \widetilde{\mathcal{L}}(f)$ holds $p \in \mathrm{LeftComp}(f)$.

(120) For every point p of $\mathcal{E}^2_{\rm T}$ such that $\pi_1 f = \text{N-min}\, \widetilde{\mathcal{L}}(f)$ and $p_{\mathbf{2}} < \text{S-bound}\, \widetilde{\mathcal{L}}(f)$ holds $p \in \mathrm{LeftComp}(f)$.

(121) For every point p of $\mathcal{E}^2_{\rm T}$ such that $\pi_1 f = \text{N-min}\, \widetilde{\mathcal{L}}(f)$ and $p_{\mathbf{2}} > \text{N-bound}\, \widetilde{\mathcal{L}}(f)$ holds $p \in \mathrm{LeftComp}(f)$.

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