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Properties of the Trigonometric Function

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Summary. This article introduces the monotone increasing and the monotone decreasing of *sinus* and *cosine*, and definitions of hyperbolic *sinus*, hyperbolic *cosine* and hyperbolic *tangent*, and some related formulas about them.

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The papers [21], [6], [17], [22], [4], [14], [15], [20], [2], [19], [3], [18], [13], [5], [7], [8], [16], [9], [10], [1], [23], [11], and [12] provide the notation and terminology for this paper.

1. Monotone Increasing and Monotone Decreasing of Sinus and Cosine

We adopt the following rules: p, q, r, t_1 are elements of $\mathbb{R}$ and n is a natural number.

Next we state a number of propositions:

(1) If $p \geqslant 0$ and $r \geqslant 0$, then $p + r \geqslant 2 \cdot \sqrt{p \cdot r}$.

(2) sin is increasing on $]0, \frac{\mathrm{Pai}}{2}[$.

(3) sin is decreasing on $]\frac{\mathrm{Pai}}{2}, \mathrm{Pai}[$.

(4) cos is decreasing on $]0, \frac{\mathrm{Pai}}{2}[$.

(5) cos is decreasing on $]\frac{\mathrm{Pai}}{2}, \mathrm{Pai}[$.

(6) sin is decreasing on $]\mathrm{Pai}, \frac{3}{2} \cdot \mathrm{Pai}[$.

(7) sin is increasing on $]\frac{3}{2} \cdot \mathrm{Pai}, 2 \cdot \mathrm{Pai}[$.

(8) cos is increasing on $]\mathrm{Pai}, \frac{3}{2} \cdot \mathrm{Pai}[$.

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(9) cos is increasing on $]\frac{3}{2} \cdot \mathrm{Pai}, 2 \cdot \mathrm{Pai}[$.

(10) $(\sin)(t_1) = (\sin)(2 \cdot \mathrm{Pai} \cdot n + t_1)$.

(11) $(\cos)(t_1) = (\cos)(2 \cdot \mathrm{Pai} \cdot n + t_1)$.

2. Hyperbolic Sinus, Hyperbolic Cosine and Hyperbolic Tangent

The partial function sinh from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:

(Def. 1) $\mathrm{dom}\,\mathrm{sinh} = \mathbb{R}$ and for every real number d holds $(\sinh)(d) = \frac{(\exp)(d) - (\exp)(-d)}{2}$.

Let d be a real number. The functor $\sinh d$ yielding an element of $\mathbb{R}$ is defined by:

(Def. 2) $\sinh d = (\sinh)(d)$.

The partial function cosh from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:

(Def. 3) $\mathrm{dom}\,\mathrm{cosh} = \mathbb{R}$ and for every real number d holds $(\cosh)(d) = \frac{(\exp)(d) + (\exp)(-d)}{2}$.

Let d be a real number. The functor $\cosh d$ yields an element of $\mathbb{R}$ and is defined as follows:

(Def. 4) $\cosh d = (\cosh)(d)$.

The partial function tanh from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:

(Def. 5) $\mathrm{dom}\,\mathrm{tanh} = \mathbb{R}$ and for every real number d holds $(\tanh)(d) = \frac{(\exp)(d) - (\exp)(-d)}{(\exp)(d) + (\exp)(-d)}$.

Let d be a real number. The functor $\tanh d$ yields an element of $\mathbb{R}$ and is defined as follows:

(Def. 6) $\tanh d = (\tanh)(d)$.

We now state a number of propositions:

(12) $(\exp)(p + q) = (\exp)(p) \cdot (\exp)(q)$.

(13) $(\exp)(0) = 1$.

(14) $(\cosh)(p)^{\mathbf{2}} - (\sinh)(p)^{\mathbf{2}} = 1$ and $(\cosh)(p) \cdot (\cosh)(p) - (\sinh)(p) \cdot (\sinh)(p) = 1$.

(15) $(\cosh)(p) \neq 0$ and $(\cosh)(p) > 0$ and $(\cosh)(0) = 1$.

(16) $(\sinh)(0) = 0$.

(17) $(\tanh)(p) = \frac{(\sinh)(p)}{(\cosh)(p)}$.

(18) $(\sinh)(p)^{\mathbf{2}} = \frac{1}{2} \cdot ((\cosh)(2 \cdot p) - 1)$ and $(\cosh)(p)^{\mathbf{2}} = \frac{1}{2} \cdot ((\cosh)(2 \cdot p) + 1)$.

(19) $(\cosh)(-p) = (\cosh)(p)$ and $(\sinh)(-p) = -(\sinh)(p)$ and $(\tanh)(-p) = -(\tanh)(p)$.

(20) $(\cosh)(p + r) = (\cosh)(p) \cdot (\cosh)(r) + (\sinh)(p) \cdot (\sinh)(r)$ and $(\cosh)(p - r) = (\cosh)(p) \cdot (\cosh)(r) - (\sinh)(p) \cdot (\sinh)(r)$.

(21) $(\sinh)(p+r) = (\sinh)(p) \cdot (\cosh)(r) + (\cosh)(p) \cdot (\sinh)(r)$ and $(\sinh)(p-r) = (\sinh)(p) \cdot (\cosh)(r) - (\cosh)(p) \cdot (\sinh)(r)$.

(22) $(\tanh)(p+r) = \frac{(\tanh)(p)+(\tanh)(r)}{1+(\tanh)(p)\cdot(\tanh)(r)}$ and $(\tanh)(p-r) = \frac{(\tanh)(p)-(\tanh)(r)}{1-(\tanh)(p)\cdot(\tanh)(r)}$.

(23) $(\sinh)(2 \cdot p) = 2 \cdot (\sinh)(p) \cdot (\cosh)(p)$ and $(\cosh)(2 \cdot p) = 2 \cdot (\cosh)(p)^{\mathbf{2}} - 1$ and $(\tanh)(2 \cdot p) = \frac{2\cdot(\tanh)(p)}{1+(\tanh)(p)^{\mathbf{2}}}$.

(24) $(\sinh)(p)^{\mathbf{2}} - (\sinh)(q)^{\mathbf{2}} = (\sinh)(p+q) \cdot (\sinh)(p-q)$ and $(\sinh)(p+q) \cdot (\sinh)(p-q) = (\cosh)(p)^{\mathbf{2}} - (\cosh)(q)^{\mathbf{2}}$ and $(\sinh)(p)^{\mathbf{2}} - (\sinh)(q)^{\mathbf{2}} = (\cosh)(p)^{\mathbf{2}} - (\cosh)(q)^{\mathbf{2}}$.

(25) $(\sinh)(p)^{\mathbf{2}} + (\cosh)(q)^{\mathbf{2}} = (\cosh)(p+q) \cdot (\cosh)(p-q)$ and $(\cosh)(p+q) \cdot (\cosh)(p-q) = (\cosh)(p)^{\mathbf{2}} + (\sinh)(q)^{\mathbf{2}}$ and $(\sinh)(p)^{\mathbf{2}} + (\cosh)(q)^{\mathbf{2}} = (\cosh)(p)^{\mathbf{2}} + (\sinh)(q)^{\mathbf{2}}$.

(26) $(\sinh)(p) + (\sinh)(r) = 2 \cdot (\sinh)(\frac{p}{2} + \frac{r}{2}) \cdot (\cosh)(\frac{p}{2} - \frac{r}{2})$ and $(\sinh)(p) - (\sinh)(r) = 2 \cdot (\sinh)(\frac{p}{2} - \frac{r}{2}) \cdot (\cosh)(\frac{p}{2} + \frac{r}{2})$.

(27) $(\cosh)(p) + (\cosh)(r) = 2 \cdot (\cosh)(\frac{p}{2} + \frac{r}{2}) \cdot (\cosh)(\frac{p}{2} - \frac{r}{2})$ and $(\cosh)(p) - (\cosh)(r) = 2 \cdot (\sinh)(\frac{p}{2} - \frac{r}{2}) \cdot (\sinh)(\frac{p}{2} + \frac{r}{2})$.

(28) $(\tanh)(p) + (\tanh)(r) = \frac{(\sinh)(p+r)}{(\cosh)(p)\cdot(\cosh)(r)}$ and $(\tanh)(p) - (\tanh)(r) = \frac{(\sinh)(p-r)}{(\cosh)(p)\cdot(\cosh)(r)}$.

(29) $((\cosh)(p) + (\sinh)(p))^n_{\mathbb{N}} = (\cosh)(n \cdot p) + (\sinh)(n \cdot p)$.

One can check the following observations:

* sinh is total,
* cosh is total, and
* tanh is total.

One can prove the following propositions:

(30) $\operatorname{dom}\sinh = \mathbb{R}$ and $\operatorname{dom}\cosh = \mathbb{R}$ and $\operatorname{dom}\tanh = \mathbb{R}$.

(31) sinh is differentiable in p and $(\sinh)'(p) = (\cosh)(p)$.

(32) cosh is differentiable in p and $(\cosh)'(p) = (\sinh)(p)$.

(33) tanh is differentiable in p and $(\tanh)'(p) = \frac{1}{(\cosh)(p)^{\mathbf{2}}}$.

(34) sinh is differentiable on $\mathbb{R}$ and $(\sinh)'(p) = (\cosh)(p)$.

(35) cosh is differentiable on $\mathbb{R}$ and $(\cosh)'(p) = (\sinh)(p)$.

(36) tanh is differentiable on $\mathbb{R}$ and $(\tanh)'(p) = \frac{1}{(\cosh)(p)^{\mathbf{2}}}$.

(37) $(\cosh)(p) \geqslant 1$.

(38) sinh is continuous in p.

(39) cosh is continuous in p.

(40) tanh is continuous in p.

(41) sinh is continuous on $\mathbb{R}$.

(42) cosh is continuous on $\mathbb{R}$.

(43) tanh is continuous on $\mathbb{R}$.

(44) $(\tanh)(p) < 1$ and $(\tanh)(p) > -1$.

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