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# Some Properties of Isomorphism between Relational Structures. On the Product of Topological Spaces[1]

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The articles [1], [12], [7], [8], [9], [10], [19], [2], [26], [14], [24], [20], [21], [28], [29], [22], [27], [23], [17], [13], [31], [6], [16], [15], [4], [11], [5], [18], [3], [30], and [25] provide the terminology and notation for this paper.

## 1. Preliminaries

The following propositions are true:

(1) $2^1 = \{0, 1\}$.

(2) For every set $X$ and for every subset $Y$ of $X$ holds $\mathrm{rng}(\mathrm{id}_X \restriction Y) = Y$.

(3) For every function $f$ and for all sets $a$, $b$ holds $(f + \cdot (a \mapsto b))(a) = b$.

Let us observe that there exists a relational structure which is strict and empty.

Next we state four propositions:

(4) Let $S$ be an empty 1-sorted structure, $T$ be a 1-sorted structure, and $f$ be a map from $S$ into $T$. If $\mathrm{rng}\, f = \Omega_T$, then $T$ is empty.

(5) Let $S$ be a 1-sorted structure, $T$ be an empty 1-sorted structure, and $f$ be a map from $S$ into $T$. If $\mathrm{dom}\, f = \Omega_S$, then $S$ is empty.

(6) Let $S$ be a non empty 1-sorted structure, $T$ be a 1-sorted structure, and $f$ be a map from $S$ into $T$. If $\mathrm{dom}\, f = \Omega_S$, then $T$ is non empty.

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(7) Let $S$ be a 1-sorted structure, $T$ be a non empty 1-sorted structure, and $f$ be a map from $S$ into $T$. If $\operatorname{rng} f = \Omega_T$, then $S$ is non empty.

Let $S$ be a non empty reflexive relational structure, let $T$ be a non empty relational structure, and let $f$ be a map from $S$ into $T$. Let us observe that $f$ is directed-sups-preserving if and only if:

(Def. 1) For every non empty directed subset $X$ of $S$ holds $f$ preserves sup of $X$.

Let $R$ be a 1-sorted structure and let $N$ be a net structure over $R$. We say that $N$ is function yielding if and only if:

(Def. 2) The mapping of $N$ is function yielding.

Let us note that there exists a 1-sorted structure which is strict, non empty, and constituted functions.

One can verify that there exists a relational structure which is strict, non empty, and constituted functions.

Let $R$ be a constituted functions 1-sorted structure. One can verify that every net structure over $R$ is function yielding.

Let $R$ be a constituted functions 1-sorted structure. Note that there exists a net structure over $R$ which is strict and function yielding.

Let $R$ be a non empty constituted functions 1-sorted structure. Note that there exists a net structure over $R$ which is strict, non empty, and function yielding.

Let $R$ be a constituted functions 1-sorted structure and let $N$ be a function yielding net structure over $R$. Observe that the mapping of $N$ is function yielding.

Let $R$ be a non empty constituted functions 1-sorted structure. Note that there exists a net in $R$ which is strict and function yielding.

Let $S$ be a non empty 1-sorted structure and let $N$ be a non empty net structure over $S$. Note that $\operatorname{rng}$ (the mapping of $N$) is non empty.

Let $S$ be a non empty 1-sorted structure and let $N$ be a non empty net structure over $S$. Observe that $\operatorname{rng} \operatorname{netmap}(N, S)$ is non empty.

One can prove the following two propositions:

(8) Let $A$, $B$, $C$ be non empty relational structures, $f$ be a map from $B$ into $C$, and $g$, $h$ be maps from $A$ into $B$. If $g \leqslant h$ and $f$ is monotone, then $f \cdot g \leqslant f \cdot h$.

(9) Let $S$ be a non empty topological space, $T$ be a non empty topological space-like FR-structure, $f$, $g$ be maps from $S$ into $T$, and $x$, $y$ be elements of $[S \to T]$. If $x = f$ and $y = g$, then $x \leqslant y$ iff $f \leqslant g$.

Let $I$ be a set and let $R$ be a non empty relational structure. Note that every element of the carrier of $R^I$ is function-like and relation-like.

Let $I$ be a non empty set, let $R$ be a non empty relational structure, let $f$ be an element of the carrier of $R^I$, and let $i$ be an element of $I$. Then $f(i)$ is an element of $R$.

## 2. Some Properties of Isomorphism between Relational Structures

One can prove the following proposition

(10) For all relational structures $S$, $T$ and for every map $f$ from $S$ into $T$ such that $f$ is isomorphic holds $f$ is onto.

Let $S$, $T$ be relational structures. Note that every map from $S$ into $T$ which is isomorphic is also onto.

We now state four propositions:

(11) Let $S$, $T$ be non empty relational structures and $f$ be a map from $S$ into $T$. If $f$ is isomorphic, then $f^{-1}$ is isomorphic.

(12) For all non empty relational structures $S$, $T$ such that $S$ and $T$ are isomorphic and $S$ has g.l.b.'s holds $T$ has g.l.b.'s.

(13) For all non empty relational structures $S$, $T$ such that $S$ and $T$ are isomorphic and $S$ has l.u.b.'s holds $T$ has l.u.b.'s.

(14) For every relational structure $L$ such that $L$ is empty holds $L$ is bounded.

Let us note that every relational structure which is empty is also bounded.

The following propositions are true:

(15) Let $S$, $T$ be relational structures. Suppose $S$ and $T$ are isomorphic and $S$ is lower-bounded. Then $T$ is lower-bounded.

(16) Let $S$, $T$ be relational structures. Suppose $S$ and $T$ are isomorphic and $S$ is upper-bounded. Then $T$ is upper-bounded.

(17) Let $S$, $T$ be non empty relational structures, $A$ be a subset of $S$, and $f$ be a map from $S$ into $T$. Suppose $f$ is isomorphic and sup $A$ exists in $S$. Then sup $f^\circ A$ exists in $T$.

(18) Let $S$, $T$ be non empty relational structures, $A$ be a subset of $S$, and $f$ be a map from $S$ into $T$. Suppose $f$ is isomorphic and inf $A$ exists in $S$. Then inf $f^\circ A$ exists in $T$.

## 3. On the Product of Topological Spaces

Next we state two propositions:

(19) Let $S$, $T$ be topological structures. Suppose $S$ and $T$ are homeomorphic or there exists a map $f$ from $S$ into $T$ such that $\mathrm{dom}\, f = \Omega_S$ and $\mathrm{rng}\, f = \Omega_T$. Then $S$ is empty if and only if $T$ is empty.

(20) For every non empty topological space $T$ holds $T$ and the topological structure of $T$ are homeomorphic.

Let $T$ be a Scott reflexive non empty FR-structure. One can verify that every subset of $T$ which is open is also inaccessible and upper and every subset of $T$ which is inaccessible and upper is also open.

Next we state several propositions:

(21) Let $T$ be a topological structure, $x$, $y$ be points of $T$, and $X$, $Y$ be subsets of $T$. If $X = \{x\}$ and $\overline{X} \subseteq \overline{Y}$, then $x \in \overline{Y}$.

(22) Let $T$ be a topological structure, $x$, $y$ be points of $T$, and $Y$, $V$ be subsets of $T$. If $Y = \{y\}$ and $x \in \overline{Y}$ and $V$ is open and $x \in V$, then $y \in V$.

(23) Let $T$ be a topological structure, $x$, $y$ be points of $T$, and $X$, $Y$ be subsets of $T$. Suppose $X = \{x\}$ and $Y = \{y\}$. Suppose that for every subset $V$ of $T$ such that $V$ is open holds if $x \in V$, then $y \in V$. Then $\overline{X} \subseteq \overline{Y}$.

(24) Let $S$, $T$ be non empty topological spaces, $A$ be an irreducible subset of $S$, and $B$ be a subset of $T$. Suppose $A = B$ and the topological structure of $S =$ the topological structure of $T$. Then $B$ is irreducible.

(25) Let $S$, $T$ be non empty topological spaces, $a$ be a point of $S$, $b$ be a point of $T$, $A$ be a subset of the carrier of $S$, and $B$ be a subset of the carrier of $T$. Suppose $a = b$ and $A = B$ and the topological structure of $S =$ the topological structure of $T$ and $a$ is dense point of $A$. Then $b$ is dense point of $B$.

(26) Let $S$, $T$ be topological structures, $A$ be a subset of $S$, and $B$ be a subset of $T$. Suppose $A = B$ and the topological structure of $S =$ the topological structure of $T$ and $A$ is compact. Then $B$ is compact.

(27) Let $S$, $T$ be non empty topological spaces. Suppose the topological structure of $S =$ the topological structure of $T$ and $S$ is sober. Then $T$ is sober.

(28) Let $S$, $T$ be non empty topological spaces. Suppose the topological structure of $S =$ the topological structure of $T$ and $S$ is locally-compact. Then $T$ is locally-compact.

(29) Let $S$, $T$ be topological structures. Suppose the topological structure of $S =$ the topological structure of $T$ and $S$ is compact. Then $T$ is compact.

Let $I$ be a non empty set, let $T$ be a non empty topological space, let $x$ be a point of $\prod(I \longmapsto T)$, and let $i$ be an element of $I$. Then $x(i)$ is an element of $T$.

The following propositions are true:

(30) Let $M$ be a non empty set, $J$ be a topological space yielding nonempty many sorted set indexed by $M$, and $x$, $y$ be points of $\prod J$. Then $x \in \overline{\{y\}}$ if and only if for every element $i$ of $M$ holds $x(i) \in \overline{\{y(i)\}}$.

(31) Let $M$ be a non empty set, $T$ be a non empty topological space, and $x$, $y$ be points of $\prod(M \longmapsto T)$. Then $x \in \overline{\{y\}}$ if and only if for every element $i$ of $M$ holds $x(i) \in \overline{\{y(i)\}}$.

(32) Let $M$ be a non empty set, $i$ be an element of $M$, $J$ be a topological

space yielding nonempty many sorted set indexed by $M$, and $x$ be a point of $\prod J$. Then $\pi_i\overline{\{x\}} = \overline{\{x(i)\}}$.

(33) Let $M$ be a non empty set, $i$ be an element of $M$, $T$ be a non empty topological space, and $x$ be a point of $\prod(M \longmapsto T)$. Then $\pi_i\overline{\{x\}} = \overline{\{x(i)\}}$.

(34) Let $X$, $Y$ be non empty topological structures, $f$ be a map from $X$ into $Y$, and $g$ be a map from $Y$ into $X$. Suppose $f = \mathrm{id}_X$ and $g = \mathrm{id}_X$ and $f$ is continuous and $g$ is continuous. Then the topological structure of $X =$ the topological structure of $Y$.

(35) Let $X$, $Y$ be non empty topological spaces and $f$ be a map from $X$ into $Y$. If $f^\circ$ is continuous, then $f$ is continuous.

Let $X$, $Y$ be non empty topological spaces. Observe that every continuous map from $X$ into $Y$ is continuous.

Let $X$ be a non empty topological space and let $Y$ be a non empty subspace of $X$. Note that $\overset{Y}{\hookrightarrow}$ is continuous.

The following propositions are true:

(36) For every non empty topological space $T$ and for every map $f$ from $T$ into $T$ such that $f \cdot f = f$ holds $f^\circ \cdot (\overset{\mathrm{Im}\, f}{\hookrightarrow}) = \mathrm{id}_{\mathrm{Im}\, f}$.

(37) For every non empty topological space $Y$ and for every non empty subspace $W$ of $Y$ holds $(\overset{W}{\hookrightarrow})^\circ$ is a homeomorphism.

(38) Let $M$ be a non empty set and $J$ be a topological space yielding nonempty many sorted set indexed by $M$. Suppose that for every element $i$ of $M$ holds $J(i)$ is a $T_0$ topological space. Then $\prod J$ is $T_0$.

Let $I$ be a non empty set and let $T$ be a non empty $T_0$ topological space. One can check that $\prod(I \longmapsto T)$ is $T_0$.

The following proposition is true

(39) Let $M$ be a non empty set and $J$ be a topological space yielding nonempty many sorted set indexed by $M$. Suppose that for every element $i$ of $M$ holds $J(i)$ is $T_1$ and topological space-like. Then $\prod J$ is a $T_1$ space.

Let $I$ be a non empty set and let $T$ be a non empty $T_1$ topological space. Observe that $\prod(I \longmapsto T)$ is $T_1$.

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