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The Concept of Fuzzy Set and Membership Function and Basic Properties of Fuzzy Set Operation

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Summary. This article introduces the fuzzy theory. At first, the definition of fuzzy set characterized by membership function is described. Next, definitions of empty fuzzy set and universal fuzzy set and basic operations of these fuzzy sets are shown. At last, exclusive sum and absolute difference which are special operation are introduced.

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The terminology and notation used in this paper have been introduced in the following articles: [8], [1], [5], [9], [3], [4], [6], [7], and [2].

1. Definition of Membership Function and Fuzzy Set

In this paper C is a non empty set and c is an element of C.

We now state the proposition

(1) $\mathrm{rng}(\chi_{C,C}) \subseteq [0,1]$.

Let us consider C. A partial function from C to $\mathbb{R}$ is said to be a membership function of C if:

(Def. 1) $\mathrm{dom}\,\mathrm{it} = C$ and $\mathrm{rng}\,\mathrm{it} \subseteq [0,1]$.

The following proposition is true

(2) $\chi_{C,C}$ is a membership function of C.

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In the sequel f, h, g, h_1 denote membership functions of C.

Let C be a non empty set and let h be a membership function of C. A set is called a FuzzySet of C, h if:

(Def. 2) It $= [\!: C, h^\circ C :\!]$.

Let C be a non empty set, let h, g be membership functions of C, let A be a FuzzySet of C, h, and let B be a FuzzySet of C, g. The predicate $A = B$ is defined as follows:

(Def. 3) For every element c of C holds $h(c) = g(c)$.

Let C be a non empty set, let h, g be membership functions of C, let A be a FuzzySet of C, h, and let B be a FuzzySet of C, g. The predicate $A \subseteq B$ is defined by:

(Def. 4) For every element c of C holds $h(c) \leqslant g(c)$.

In the sequel A denotes a FuzzySet of C, f, B denotes a FuzzySet of C, g, D denotes a FuzzySet of C, h, and D_1 denotes a FuzzySet of C, h_1.

One can prove the following propositions:

(3) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

(4) $A \subseteq A$.

(5) If $A \subseteq B$ and $B \subseteq D$, then $A \subseteq D$.

2. Intersection, Union and Complement

Let C be a non empty set and let h, g be membership functions of C. The functor $\min(h, g)$ yielding a membership function of C is defined by:

(Def. 5) For every element c of C holds $(\min(h, g))(c) = \min(h(c), g(c))$.

Let C be a non empty set and let h, g be membership functions of C. The functor $\max(h, g)$ yields a membership function of C and is defined by:

(Def. 6) For every element c of C holds $(\max(h, g))(c) = \max(h(c), g(c))$.

Let C be a non empty set and let h be a membership function of C. The functor 1-minus h yielding a membership function of C is defined by:

(Def. 7) For every element c of C holds $(\text{1-minus}\, h)(c) = 1 - h(c)$.

Let C be a non empty set, let h, g be membership functions of C, let A be a FuzzySet of C, h, and let B be a FuzzySet of C, g. The functor $A \cap B$ yielding a FuzzySet of C, $\min(h, g)$ is defined as follows:

(Def. 8) $A \cap B = [\!: C, (\min(h, g))^\circ C :\!]$.

Let C be a non empty set, let h, g be membership functions of C, let A be a FuzzySet of C, h, and let B be a FuzzySet of C, g. The functor $A \cup B$ yields a FuzzySet of C, $\max(h, g)$ and is defined by:

(Def. 9) $A \cup B = [\!: C, (\max(h, g))^\circ C :\!]$.

Let C be a non empty set, let h be a membership function of C, and let A be a FuzzySet of C, h. The functor $A^{\rm c}$ yielding a FuzzySet of C, 1-minus h is defined by:

(Def. 10) $A^{\rm c} = [\!: C, (\text{1-minus}\, h)^{\circ} C :\!]$.

We now state a number of propositions:

(6) $\min(h(c), g(c)) = (\min(h, g))(c)$ and $\max(h(c), g(c)) = (\max(h, g))(c)$.

(7) $\max(h, h) = h$ and $\min(h, h) = h$ and $\max(h, h) = \min(h, h)$ and $\min(f, g) = \min(g, f)$ and $\max(f, g) = \max(g, f)$.

(8) $f = g$ iff $A = B$.

(9) $A \cap A = A$ and $A \cup A = A$.

(10) $A \cap B = B \cap A$ and $A \cup B = B \cup A$.

(11) $\max(\max(f, g), h) = \max(f, \max(g, h))$ and $\min(\min(f, g), h) = \min(f, \min(g, h))$.

(12) $(A \cup B) \cup D = A \cup (B \cup D)$.

(13) $(A \cap B) \cap D = A \cap (B \cap D)$.

(14) $\max(f, \min(f, g)) = f$ and $\min(f, \max(f, g)) = f$.

(15) $A \cup A \cap B = A$ and $A \cap (A \cup B) = A$.

(16) $\min(f, \max(g, h)) = \max(\min(f, g), \min(f, h))$ and $\max(f, \min(g, h)) = \min(\max(f, g), \max(f, h))$.

(17) $A \cup B \cap D = (A \cup B) \cap (A \cup D)$ and $A \cap (B \cup D) = A \cap B \cup A \cap D$.

(18) 1-minus 1-minus $h = h$.

(19) $(A^{\rm c})^{\rm c} = A$.

(20) 1-minus $\max(f, g) = \min(\text{1-minus}\, f, \text{1-minus}\, g)$ and 1-minus $\min(f, g) = \max(\text{1-minus}\, f, \text{1-minus}\, g)$.

(21) $(A \cup B)^{\rm c} = A^{\rm c} \cap B^{\rm c}$ and $(A \cap B)^{\rm c} = A^{\rm c} \cup B^{\rm c}$.

3. Empty Fuzzy Set and Universal Fuzzy Set

Let C be a non empty set. A set is called an Empty FuzzySet of C if:

(Def. 11) It $= [\!: C, (\chi_{\emptyset,C})^{\circ} C :\!]$.

Let C be a non empty set. A set is called a Universal FuzzySet of C if:

(Def. 12) It $= [\!: C, (\chi_{C,C})^{\circ} C :\!]$.

In the sequel X is a Universal FuzzySet of C and E is an Empty FuzzySet of C.

One can prove the following two propositions:

(22) $\operatorname{rng}(\chi_{\emptyset,C}) \subseteq [0, 1]$.

(23) $\chi_{\emptyset,C}$ is a membership function of C.

Let C be a non empty set. The functor $\mathrm{EMF}\,C$ yields a membership function of C and is defined as follows:

(Def. 13) $\mathrm{EMF}\,C = \chi_{\emptyset,C}$.

Let C be a non empty set. The functor $\mathrm{UMF}\,C$ yields a membership function of C and is defined as follows:

(Def. 14) $\mathrm{UMF}\,C = \chi_{C,C}$.

One can prove the following propositions:

(24) For every membership function h of C such that $h = \chi_{C,C}$ holds $[\!: C, (\chi_{C,C})^{\circ}C :\!]$ is a FuzzySet of C, h.

(25) For every membership function h of C such that $h = \chi_{\emptyset,C}$ holds $[\!: C, (\chi_{\emptyset,C})^{\circ}C :\!]$ is a FuzzySet of C, h.

(26) E is a FuzzySet of C, $\mathrm{EMF}\,C$.

(27) X is a FuzzySet of C, $\mathrm{UMF}\,C$.

Let C be a non empty set. We see that the Empty FuzzySet of C is a FuzzySet of C, $\mathrm{EMF}\,C$.

Let C be a non empty set. We see that the Universal FuzzySet of C is a FuzzySet of C, $\mathrm{UMF}\,C$.

In the sequel X denotes a Universal FuzzySet of C and E denotes an Empty FuzzySet of C.

One can prove the following propositions:

(28) Let a, b be elements of $\mathbb{R}$ and f be a partial function from C to $\mathbb{R}$. Suppose $\mathrm{rng}\, f \subseteq [a, b]$ and $\mathrm{dom}\, f \neq \emptyset$ and $a \leqslant b$. Let x be an element of C. If $x \in \mathrm{dom}\, f$, then $a \leqslant f(x)$ and $f(x) \leqslant b$.

(29) $E \subseteq A$.

(30) $A \subseteq X$.

(31) For every element x of C and for every membership function h of C holds $(\mathrm{EMF}\,C)(x) \leqslant h(x)$ and $h(x) \leqslant (\mathrm{UMF}\,C)(x)$.

(32) $\max(f, \mathrm{UMF}\,C) = \mathrm{UMF}\,C$ and $\min(f, \mathrm{UMF}\,C) = f$ and $\max(f, \mathrm{EMF}\,C) = f$ and $\min(f, \mathrm{EMF}\,C) = \mathrm{EMF}\,C$.

(33) $A \cup X = X$ and $A \cap X = A$.

(34) $A \cup E = A$ and $A \cap E = E$.

(35) $A \subseteq A \cup B$.

(36) If $A \subseteq D$ and $B \subseteq D$, then $A \cup B \subseteq D$.

(37) For all elements a, b, c of $\mathbb{R}$ such that $a \leqslant b$ holds $\max(a, c) \leqslant \max(b, c)$.

(38) If $A \subseteq B$, then $A \cup D \subseteq B \cup D$.

(39) If $A \subseteq B$ and $D \subseteq D_1$, then $A \cup D \subseteq B \cup D_1$.

(40) If $A \subseteq B$, then $A \cup B = B$.

(41) $A \cap B \subseteq A$.

(42) $A \cap B \subseteq A \cup B$.

(43) If $D \subseteq A$ and $D \subseteq B$, then $D \subseteq A \cap B$.

(44) For all elements a, b, c, d of $\mathbb{R}$ such that $a \leqslant b$ and $c \leqslant d$ holds $\min(a, c) \leqslant \min(b, d)$.

(45) For all elements a, b, c of $\mathbb{R}$ such that $a \leqslant b$ holds $\min(a, c) \leqslant \min(b, c)$.

(46) If $A \subseteq B$, then $A \cap D \subseteq B \cap D$.

(47) If $A \subseteq B$ and $D \subseteq D_1$, then $A \cap D \subseteq B \cap D_1$.

(48) If $A \subseteq B$, then $A \cap B = A$.

(49) If $A \subseteq B$ and $A \subseteq D$ and $B \cap D = E$, then $A = E$.

(50) If $A \cap B \cup A \cap D = A$, then $A \subseteq B \cup D$.

(51) If $A \subseteq B$ and $B \cap D = E$, then $A \cap D = E$.

(52) If $A \subseteq E$, then $A = E$.

(53) $A \cup B = E$ iff $A = E$ and $B = E$.

(54) $A = B \cup D$ iff $B \subseteq A$ and $D \subseteq A$ and for all h_1, D_1 such that $B \subseteq D_1$ and $D \subseteq D_1$ holds $A \subseteq D_1$.

(55) $A = B \cap D$ iff $A \subseteq B$ and $A \subseteq D$ and for all h_1, D_1 such that $D_1 \subseteq B$ and $D_1 \subseteq D$ holds $D_1 \subseteq A$.

(56) If $A \subseteq B \cup D$ and $A \cap D = E$, then $A \subseteq B$.

(57) $A \subseteq B$ iff $B^{\mathrm{c}} \subseteq A^{\mathrm{c}}$.

(58) If $A \subseteq B^{\mathrm{c}}$, then $B \subseteq A^{\mathrm{c}}$.

(59) If $A^{\mathrm{c}} \subseteq B$, then $B^{\mathrm{c}} \subseteq A$.

(60) $(A \cup B)^{\mathrm{c}} \subseteq A^{\mathrm{c}}$ and $(A \cup B)^{\mathrm{c}} \subseteq B^{\mathrm{c}}$.

(61) $A^{\mathrm{c}} \subseteq (A \cap B)^{\mathrm{c}}$ and $B^{\mathrm{c}} \subseteq (A \cap B)^{\mathrm{c}}$.

(62) $\text{1-minus}\,\mathrm{EMF}\,C = \mathrm{UMF}\,C$ and $\text{1-minus}\,\mathrm{UMF}\,C = \mathrm{EMF}\,C$.

(63) $E^{\mathrm{c}} = X$ and $X^{\mathrm{c}} = E$.

4. Exclusive Sum, Absolute Difference

Let C be a non empty set, let h, g be membership functions of C, let A be a FuzzySet of C, h, and let B be a FuzzySet of C, g. The functor $A \dot{-} B$ yields a FuzzySet of C, $\max(\min(h, \text{1-minus}\,g), \min(\text{1-minus}\,h, g))$ and is defined as follows:

(Def. 15) $A \dot{-} B = [\!: C, (\max(\min(h, \text{1-minus}\,g), \min(\text{1-minus}\,h, g)))^{\circ}C :\!]$.

The following propositions are true:

(64) $A \dot{-} B = A \cap B^{\mathrm{c}} \cup A^{\mathrm{c}} \cap B$.

(65) $A \dot{-} B = B \dot{-} A$.

(66) $A \dot{-} E = A$ and $E \dot{-} A = A$.

(67) $A \dot{-} X = A^{\mathrm{c}}$ and $X \dot{-} A = A^{\mathrm{c}}$.

(68) $A \cap B \cup B \cap D \cup D \cap A = (A \cup B) \cap (B \cup D) \cap (D \cup A)$.

(69) $A \cap B \cup A^{\mathrm{c}} \cap B^{\mathrm{c}} \subseteq (A \dot{-} B)^{\mathrm{c}}$.

(70) $(A \dot{-} B) \cup A \cap B \subseteq A \cup B$.

(71) $A \dot{-} A = A \cap A^{\mathrm{c}}$.

Let C be a non empty set and let h, g be membership functions of C. The functor $|h - g|$ yields a membership function of C and is defined as follows:

(Def. 16) For every element c of C holds $|h - g|(c) = |h(c) - g(c)|$.

Let C be a non empty set, let h, g be membership functions of C, let A be a FuzzySet of C, h, and let B be a FuzzySet of C, g. The functor $|A - B|$ yielding a FuzzySet of C, $|h - g|$ is defined by:

(Def. 17) $|A - B| = [\!: C, |h - g|^{\circ} C :\!]$.

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