FORMALIZED MATHEMATICS
Volume 9, Number 2, 2001
University of Białystok

Introduction to Several Concepts of Convexity and Semicontinuity for Function from $\mathbb{R}$ to $\mathbb{R}$

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Summary. This article is an introduction to convex analysis. In the beginning, we have defined the concept of strictly convexity and proved some basic properties between convexity and strictly convexity. Moreover, we have defined concepts of other convexity and semicontinuity, and proved their basic properties.

MML Identifier: `RFUNCT_4`.

The papers [12], [3], [1], [4], [5], [9], [6], [13], [8], [16], [17], [11], [7], [10], [14], [15], and [2] provide the notation and terminology for this paper.

1. Some Useful Properties of n-Tuples on $\mathbb{R}$

We adopt the following convention: a, b, r, s, x_0, x are real numbers, f, g are partial functions from $\mathbb{R}$ to $\mathbb{R}$, and X, Y are sets.

The following propositions are true:

(1) $\max(a,b) \geqslant \min(a,b)$.

(2) Let n be a natural number, R_1, R_2 be elements of $\mathbb{R}^n$, and r_1, r_2 be real numbers. Then $R_1 \frown \langle r_1 \rangle \bullet R_2 \frown \langle r_2 \rangle = (R_1 \bullet R_2) \frown \langle r_1 \cdot r_2 \rangle$.

(3) Let n be a natural number and R be an element of $\mathbb{R}^n$. Suppose $\sum R = 0$ and for every natural number i such that $i \in \operatorname{dom} R$ holds $0 \leqslant R(i)$. Let i be a natural number. If $i \in \operatorname{dom} R$, then $R(i) = 0$.

ISSN 1426–2630

(4) Let n be a natural number and R be an element of $\mathbb{R}^n$. Suppose that for every natural number i such that $i \in \operatorname{dom} R$ holds $0 = R(i)$. Then $R = n \mapsto (0$ **qua** real number).

(5) For every natural number n and for every element R of $\mathbb{R}^n$ holds $n \mapsto (0$ **qua** real number) $\bullet R = n \mapsto (0$ **qua** real number).

2. Convex and Strictly Convex Functions

Let us consider f, X. We say that f is strictly convex on X if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every real number p such that $0 < p$ and $p < 1$ and for all real numbers r, s such that $r \in X$ and $s \in X$ and $p \cdot r + (1-p) \cdot s \in X$ and $r \neq s$ holds $f(p \cdot r + (1-p) \cdot s) < p \cdot f(r) + (1-p) \cdot f(s)$.

We now state a number of propositions:

(6) If f is strictly convex on X, then f is convex on X.

(7) Let a, b be real numbers and f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Then f is strictly convex on $[a, b]$ if and only if the following conditions are satisfied:

(i) $[a, b] \subseteq \operatorname{dom} f$, and

(ii) for every real number p such that $0 < p$ and $p < 1$ and for all real numbers r, s such that $r \in [a, b]$ and $s \in [a, b]$ and $r \neq s$ holds $f(p \cdot r + (1-p) \cdot s) < p \cdot f(r) + (1-p) \cdot f(s)$.

(8) Let X be a set and f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Then f is convex on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for all real numbers a, b, c such that $a \in X$ and $b \in X$ and $c \in X$ and $a < b$ and $b < c$ holds $f(b) \leqslant \frac{c-b}{c-a} \cdot f(a) + \frac{b-a}{c-a} \cdot f(c)$.

(9) Let X be a set and f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Then f is strictly convex on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for all real numbers a, b, c such that $a \in X$ and $b \in X$ and $c \in X$ and $a < b$ and $b < c$ holds $f(b) < \frac{c-b}{c-a} \cdot f(a) + \frac{b-a}{c-a} \cdot f(c)$.

(10) If f is strictly convex on X and $Y \subseteq X$, then f is strictly convex on Y.

(11) f is strictly convex on X iff $f - r$ is strictly convex on X.

(12) If $0 < r$, then f is strictly convex on X iff $r\, f$ is strictly convex on X.

(13) If f is strictly convex on X and g is strictly convex on X, then $f + g$ is strictly convex on X.

(14) If f is strictly convex on X and g is convex on X, then $f+g$ is strictly convex on X.

(15) Suppose f is strictly convex on X but g is strictly convex on X but $a>0$ and $b \geqslant 0$ or $a \geqslant 0$ and $b>0$. Then $a\,f+b\,g$ is strictly convex on X.

(16) f is convex on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for all a, b, r such that $a \in X$ and $b \in X$ and $r \in X$ and $a<r$ and $r<b$ holds $\frac{f(r)-f(a)}{r-a} \leqslant \frac{f(b)-f(a)}{b-a}$ and $\frac{f(b)-f(a)}{b-a} \leqslant \frac{f(b)-f(r)}{b-r}$.

(17) f is strictly convex on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for all a, b, r such that $a \in X$ and $b \in X$ and $r \in X$ and $a<r$ and $r<b$ holds $\frac{f(r)-f(a)}{r-a} < \frac{f(b)-f(a)}{b-a}$ and $\frac{f(b)-f(a)}{b-a} < \frac{f(b)-f(r)}{b-r}$.

(18) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is total. Then for every natural number n and for all elements P, E, F of $\mathbb{R}^n$ such that $\sum P=1$ and for every natural number i such that $i \in \operatorname{dom} P$ holds $P(i) \geqslant 0$ and $F(i)=f(E(i))$ holds $f(\sum(P \bullet E)) \leqslant \sum(P \bullet F)$ if and only if f is convex on $\mathbb{R}$.

(19) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, I be an interval, and a be a real number. Suppose there exist real numbers x_1, x_2 such that $x_1 \in I$ and $x_2 \in I$ and $x_1<a$ and $a<x_2$ and f is convex on I. Then f is continuous in a.

3. Definitions of Several Convexity and Semicontinuity Concepts

Let us consider f, X. We say that f is quasiconvex on X if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every real number p such that $0<p$ and $p<1$ and for all real numbers r, s such that $r \in X$ and $s \in X$ and $p \cdot r+(1-p) \cdot s \in X$ holds $f(p \cdot r+(1-p) \cdot s) \leqslant \max(f(r), f(s))$.

Let us consider f, X. We say that f is strictly quasiconvex on X if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every real number p such that $0<p$ and $p<1$ and for all real numbers r, s such that $r \in X$ and $s \in X$ and $p \cdot r+(1-p) \cdot s \in X$ and $f(r) \neq f(s)$ holds $f(p \cdot r+(1-p) \cdot s)<\max(f(r), f(s))$.

Let us consider f, X. We say that f is strongly quasiconvex on X if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every real number p such that $0 < p$ and $p < 1$ and for all real numbers r, s such that $r \in X$ and $s \in X$ and $p \cdot r + (1-p) \cdot s \in X$ and $r \neq s$ holds $f(p \cdot r + (1-p) \cdot s) < \max(f(r), f(s))$.

Let us consider f, x_0. We say that f is upper semicontinuous in x_0 if and only if:

(Def. 5) $x_0 \in \operatorname{dom} f$ and for every r such that $0 < r$ there exists s such that $0 < s$ and for every x such that $x \in \operatorname{dom} f$ and $|x - x_0| < s$ holds $f(x_0) - f(x) < r$.

Let us consider f, X. We say that f is upper semicontinuous on X if and only if:

(Def. 6) $X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f \restriction X$ is upper semicontinuous in x_0.

Let us consider f, x_0. We say that f is lower semicontinuous in x_0 if and only if:

(Def. 7) $x_0 \in \operatorname{dom} f$ and for every r such that $0 < r$ there exists s such that $0 < s$ and for every x such that $x \in \operatorname{dom} f$ and $|x - x_0| < s$ holds $f(x) - f(x_0) < r$.

Let us consider f, X. We say that f is lower semicontinuous on X if and only if:

(Def. 8) $X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f \restriction X$ is lower semicontinuous in x_0.

The following propositions are true:

(20) Let given x_0, f. Then f is upper semicontinuous in x_0 and f is lower semicontinuous in x_0 if and only if f is continuous in x_0.

(21) Let given X, f. Then f is upper semicontinuous on X and f is lower semicontinuous on X if and only if f is continuous on X.

(22) For all X, f such that f is strictly convex on X holds f is strongly quasiconvex on X.

(23) For all X, f such that f is strongly quasiconvex on X holds f is quasiconvex on X.

(24) For all X, f such that f is convex on X holds f is strictly quasiconvex on X.

(25) For all X, f such that f is strongly quasiconvex on X holds f is strictly quasiconvex on X.

(26) Let given X, f. Suppose f is strictly quasiconvex on X and f is one-to-one. Then f is strongly quasiconvex on X.

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Received March 23, 2000