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Quick Sort on SCMPDS[1]

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Summary. Proving the correctness of quick sort is much more complicated than proving the correctness of the insert sort. Its difficulty is determined mainly by the following points:

- Quick sort needs to use a push-down stack.
- It contains three nested loops.
- A subroutine of this algorithm, "Partition", has no loop-invariant.

This means we cannot justify the correctness of the "Partition" subroutine by the Hoare's axiom on program verification. This article is organized as follows. First, we present several fundamental properties of "while" program and finite sequence. Second, we define the "Partition" subroutine on SCMPDS, the task of which is to split a sequence into a smaller and a larger subsequence. The definition of quick sort on SCMPDS follows. Finally, we describe the basic property of the "Partition" and quick sort, and prove their correctness.

MML Identifier: `SCPQSORT`.

The terminology and notation used here have been introduced in the following articles: [18], [19], [23], [21], [1], [3], [4], [6], [24], [2], [15], [26], [17], [11], [7], [10], [8], [9], [12], [14], [5], [13], [20], [25], [22], and [16].

1. The Several Properties of "while" Program and Finite Sequence

In this paper n, p_0 denote natural numbers.

Let I, J be shiftable Program-blocks, let a be an Int position, and let k_1 be an integer. Observe that **if** $a > k_1$ **then** I **else** J is shiftable.

Next we state the proposition

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(1) Let s be a state of SCMPDS, I be a No-StopCode shiftable Program-block, J be a shiftable Program-block, a, b be Int positions, and k_1 be an integer. Suppose $s(\mathrm{DataLoc}(s(a), k_1)) > 0$ and I is closed on s and halting on s. Then $(\mathrm{IExec}(\mathbf{if}\ a > k_1\ \mathbf{then}\ I\ \mathbf{else}\ J, s))(b) = (\mathrm{IExec}(I, s))(b)$.

One can prove the following propositions:

(2) Let s, s_1 be states of SCMPDS, I be a No-StopCode shiftable Program-block, a be an Int position, i be an integer, and m be a natural number. Suppose $\mathrm{card}\, I > 0$ and I is closed on s and halting on s and $s(\mathrm{DataLoc}(s(a), i)) > 0$ and $m = \mathrm{LifeSpan}(s{+\cdot}\,\mathrm{Initialized}(\mathrm{stop}\, I)) + 2$ and $s_1 = (\mathrm{Computation}(s{+\cdot}\,\mathrm{Initialized}(\mathrm{stop\ while} > 0(a, i, I))))(m)$. Then $s_1{\restriction}\text{Data-Loc}_{\mathrm{SCM}} = \mathrm{IExec}(I, s){\restriction}\text{Data-Loc}_{\mathrm{SCM}}$ and $s_1{+\cdot}\,\mathrm{Initialized}(\mathrm{stop\ while} > 0(a, i, I)) = s_1$.

(3) Let s be a state of SCMPDS and I be a Program-block. Suppose that for every state t of SCMPDS such that $t{\restriction}\text{Data-Loc}_{\mathrm{SCM}} = s{\restriction}\text{Data-Loc}_{\mathrm{SCM}}$ holds I is halting on t. Then I is closed on s.

(4) For all instructions i_1, i_2, i_3, i_4 of SCMPDS holds $\mathrm{card}(i_1;\, i_2;\, i_3;\, i_4) = 4$.

(5) Let s be a state of SCMPDS, I be a No-StopCode shiftable Program-block, a, x, y be Int positions, and i, c be integers. Suppose that

(i) $\mathrm{card}\, I > 0$,

(ii) $s(x) \geqslant c + s(\mathrm{DataLoc}(s(a), i))$, and

(iii) for every state t of SCMPDS such that $t(x) \geqslant c + t(\mathrm{DataLoc}(s(a), i))$ and $t(y) = s(y)$ and $t(a) = s(a)$ and $t(\mathrm{DataLoc}(s(a), i)) > 0$ holds $(\mathrm{IExec}(I, t))(a) = t(a)$ and I is closed on t and halting on t and $(\mathrm{IExec}(I, t))(\mathrm{DataLoc}(s(a), i)) < t(\mathrm{DataLoc}(s(a), i))$ and $(\mathrm{IExec}(I, t))(x) \geqslant c{+}(\mathrm{IExec}(I, t))(\mathrm{DataLoc}(s(a), i))$ and $(\mathrm{IExec}(I, t))(y) = t(y)$.

Then while $> 0(a, i, I)$ is closed on s and while $> 0(a, i, I)$ is halting on s and if $s(\mathrm{DataLoc}(s(a), i)) > 0$, then $\mathrm{IExec}(\mathrm{while} > 0(a, i, I), s) = \mathrm{IExec}(\mathrm{while} > 0(a, i, I), \mathrm{IExec}(I, s))$.

(6) Let s be a state of SCMPDS, I be a No-StopCode shiftable Program-block, a, x, y be Int positions, and i, c be integers. Suppose that

(i) $\mathrm{card}\, I > 0$,

(ii) $s(x) \geqslant c$, and

(iii) for every state t of SCMPDS such that $t(x) \geqslant c$ and $t(y) = s(y)$ and $t(a) = s(a)$ and $t(\mathrm{DataLoc}(s(a), i)) > 0$ holds $(\mathrm{IExec}(I, t))(a) = t(a)$ and I is closed on t and halting on t and $(\mathrm{IExec}(I, t))(\mathrm{DataLoc}(s(a), i)) < t(\mathrm{DataLoc}(s(a), i))$ and $(\mathrm{IExec}(I, t))(x) \geqslant c$ and $(\mathrm{IExec}(I, t))(y) = t(y)$.

Then while $> 0(a, i, I)$ is closed on s and while $> 0(a, i, I)$ is halting on s and if $s(\mathrm{DataLoc}(s(a), i)) > 0$, then $\mathrm{IExec}(\mathrm{while} > 0(a, i, I), s) = \mathrm{IExec}(\mathrm{while} > 0(a, i, I), \mathrm{IExec}(I, s))$.

(7) Let s be a state of SCMPDS, I be a No-StopCode shiftable Program-block, a, x_1, x_2, x_3, x_4 be Int positions, and i, c, m_1 be integers. Suppose that

(i) $\operatorname{card} I > 0$,

(ii) $s(x_4) = (s(x_3) - c) + s(x_1)$,

(iii) $m_1 \leqslant s(x_3) - c$, and

(iv) for every state t of SCMPDS such that $t(x_4) = (t(x_3) - c) + t(x_1)$ and $m_1 \leqslant t(x_3) - c$ and $t(x_2) = s(x_2)$ and $t(a) = s(a)$ and $t(\mathrm{DataLoc}(s(a), i)) > 0$ holds $(\mathrm{IExec}(I, t))(a) = t(a)$ and I is closed on t and halting on t and $(\mathrm{IExec}(I, t))(\mathrm{DataLoc}(s(a), i)) < t(\mathrm{DataLoc}(s(a), i))$ and $(\mathrm{IExec}(I, t))(x_4) = ((\mathrm{IExec}(I, t))(x_3) - c) + (\mathrm{IExec}(I, t))(x_1)$ and $m_1 \leqslant (\mathrm{IExec}(I, t))(x_3) - c$ and $(\mathrm{IExec}(I, t))(x_2) = t(x_2)$.

Then while $> 0(a, i, I)$ is closed on s and while $> 0(a, i, I)$ is halting on s and if $s(\mathrm{DataLoc}(s(a), i)) > 0$, then $\mathrm{IExec}(\text{while} > 0(a, i, I), s) = \mathrm{IExec}(\text{while} > 0(a, i, I), \mathrm{IExec}(I, s))$.

(8) Let f be a finite sequence of elements of $\mathbb{Z}$ and m, k_1, k, n be natural numbers. Suppose that $m \leqslant k$ and $k \leqslant n$ and $k_1 = k - 1$ and f is non decreasing on m, k_1 and f is non decreasing on $k + 1$, n and for every natural number i such that $m \leqslant i$ and $i < k$ holds $f(i) \leqslant f(k)$ and for every natural number i such that $k < i$ and $i \leqslant n$ holds $f(k) \leqslant f(i)$. Then f is non decreasing on m, n.

(9) Let f, g be finite sequences and x be arbitrary. Suppose $x \in \operatorname{dom} g$ and f and g are fiberwise equipotent. Then there exists arbitrary y such that $y \in \operatorname{dom} g$ and $f(x) = g(y)$.

(10) Let f, g, h be finite sequences. Then f and g are fiberwise equipotent if and only if $h \frown f$ and $h \frown g$ are fiberwise equipotent.

(11) Let f, g be finite sequences and m, n, j be natural numbers. Suppose that f and g are fiberwise equipotent and $m \leqslant n$ and $n \leqslant \operatorname{len} f$ and for every natural number i such that $1 \leqslant i$ and $i \leqslant m$ holds $f(i) = g(i)$ and for every natural number i such that $n < i$ and $i \leqslant \operatorname{len} f$ holds $f(i) = g(i)$ and $m < j$ and $j \leqslant n$. Then there exists a natural number k such that $m < k$ and $k \leqslant n$ and $f(j) = g(k)$.

2. Program Partition is to Split a Sequence into a Smaller and a Larger Subsequence

The Program-block Partition is defined by the condition (Def. 1).

(Def. 1) $\mathrm{Partition} = ((\mathrm{GBP}, 5) := (\mathrm{GBP}, 4))$; $\mathrm{SubFrom}(\mathrm{GBP}, 5, \mathrm{GBP}, 2)$; $((\mathrm{GBP}, 3) := (\mathrm{GBP}, 2))$; $\mathrm{AddTo}(\mathrm{GBP}, 3, 1)$; $\text{while} > 0(\mathrm{GBP}, 5, \text{while} > 0(\mathrm{GBP}, 5, ((\mathrm{GBP}, 7) := (\mathrm{GBP}, 5))$; $\mathrm{AddTo}(\mathrm{GBP}, 5, -1)$; $((\mathrm{GBP}, 6) := (\mathrm{intpos}\, 4, 0))$; $\mathrm{SubFrom}(\mathrm{GBP}, 6, \mathrm{intpos}\, 2, 0)$; (**if** $\mathrm{GBP} > 6$ **then**

AddTo(GBP, 4, −1); AddTo(GBP, 7, −1) **else** Load((GBP)$_5$:=0))); while > 0(GBP, 7, ((GBP, 5) := (GBP, 7)); AddTo(GBP, 7, −1); ((GBP, 6) := (intpos 2, 0)); SubFrom(GBP, 6, intpos 3, 0); (**if** GBP > 6 **then** AddTo(GBP, 3, 1); AddTo(GBP, 5, −1) **else** Load((GBP)$_7$:=0))); (**if** GBP > 0 **then** 5 **else** (((GBP, 6) := (intpos 4, 0)); ((intpos 4, 0) := (intpos 3, 0)); ((intpos 3, 0) := (GBP, 6)); AddTo(GBP, 5, −2); AddTo(GBP, 3, 1); AddTo(GBP, 4, −1)))); ((GBP, 6) := (intpos 4, 0)); ((intpos 4, 0) := (intpos 2, 0)); ((intpos 2, 0) := (GBP, 6)).

3. The Construction of Quick Sort

Let n, p_0 be natural numbers. The functor QuickSort(n, p_0) yielding a Program-block is defined by the condition (Def. 2).

(Def. 2) QuickSort(n, p_0) = (GBP :=0); (SBP :=1); ((SBP)$_{p_1}$:=$p_0 + 1$); ((SBP)$_{p_1+1}$:=p_1); while > 0(GBP, 1, ((GBP, 2) := (SBP, $p_1 + 1$)); SubFrom(GBP, 2, SBP, p_1); (**if** GBP > 2 **then** ((GBP, 2) := (SBP, p_1)); ((GBP, 4) := (SBP, $p_1 + 1$)); Partition; (((SBP, $p_1 + 3$) := (SBP, $p_1 + 1$)); ((SBP, $p_1 + 1$) := (GBP, 4)); ((SBP, $p_1 + 2$) := (GBP, 4)); AddTo(SBP, $p_1 + 1$, −1); AddTo(SBP, $p_1 + 2$, 1); AddTo(GBP, 1, 2)) **else** Load(AddTo(GBP, 1, −2)))), where $p_1 = p_0 + n$.

4. The Basic Property of Partition Program

The following four propositions are true:

(12) card Partition = 38.

(13) Let s be a state of SCMPDS and m_1, p_0 be natural numbers. Suppose s(GBP) = 0 and s(intpos 4) − s(intpos 2) > 0 and s(intpos 2) = m_1 and $m_1 \geqslant p_0 + 1$ and $p_0 \geqslant 7$. Then Partition is closed on s and Partition is halting on s.

(14) Let s be a state of SCMPDS, m_1, p_0, n be natural numbers, and f, f_1 be finite sequences of elements of $\mathbb{Z}$. Suppose that s(GBP) = 0 and s(intpos 4) − s(intpos 2) > 0 and s(intpos 2) = m_1 and $m_1 \geqslant p_0 + 1$ and s(intpos 4) $\leqslant p_0 + n$ and $p_0 \geqslant 7$ and f is FinSequence on s, p_0 and len $f = n$ and f_1 is FinSequence on IExec(Partition, s), p_0 and len $f_1 = n$. Then

(i) (IExec(Partition, s))(GBP) = 0,

(ii) (IExec(Partition, s))(intpos 1) = s(intpos 1),

(iii) f and f_1 are fiberwise equipotent, and

(iv) there exists a natural number m_4 such that m_4 = (IExec(Partition, s))

(intpos 4) and $m_1 \leqslant m_4$ and $m_4 \leqslant s(\text{intpos } 4)$ and for every natural number i such that $m_1 \leqslant i$ and $i < m_4$ holds $(\text{IExec}(\text{Partition}, s))(\text{intpos } m_4) \geqslant (\text{IExec}(\text{Partition}, s))(\text{intpos } i)$ and for every natural number i such that $m_4 < i$ and $i \leqslant s(\text{intpos } 4)$ holds $(\text{IExec}(\text{Partition}, s))(\text{intpos } m_4) \leqslant (\text{IExec}(\text{Partition}, s))(\text{intpos } i)$ and for every natural number i such that $i \geqslant p_0 + 1$ but $i < s(\text{intpos } 2)$ or $i > s(\text{intpos } 4)$ holds $(\text{IExec}(\text{Partition}, s))(\text{intpos } i) = s(\text{intpos } i)$.

(15) Partition is No-StopCode and shiftable.

5. The Basic Property of Quick Sort and Its Correctness

One can prove the following three propositions:

(16) $\text{card QuickSort}(n, p_0) = 57$.

(17) For all natural numbers p_0, n such that $p_0 \geqslant 7$ holds $\text{QuickSort}(n, p_0)$ is parahalting.

(18) Let s be a state of SCMPDS and p_0, n be natural numbers. Suppose $p_0 \geqslant 7$. Then there exist finite sequences f, g of elements of $\mathbb{Z}$ such that

(i) $\text{len } f = n$,

(ii) f is FinSequence on s, p_0,

(iii) $\text{len } g = n$,

(iv) g is FinSequence on $\text{IExec}(\text{QuickSort}(n, p_0), s)$, p_0,

(v) f and g are fiberwise equipotent, and

(vi) g is non decreasing on 1, n.

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