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On the Order-consistent Topology of Complete and Uncomplete Lattices

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Summary. This paper is a continuation of the formalisation of [5] pp. 108–109. Order-consistent and upper topologies are defined. The theorem that the Scott and the upper topologies are order-consistent is proved. Remark 1.4 and example 1.5(2) are generalized for proving this theorem.

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The terminology and notation used in this paper are introduced in the following papers: [8], [12], [1], [13], [9], [15], [14], [16], [11], [3], [6], [7], [2], [10], and [4].

Let T be a non empty FR-structure. We say that T is upper if and only if:

(Def. 1) $\{-\downarrow x : x$ ranges over elements of $T\}$ is a prebasis of T.

Let us mention that there exists a top-lattice which is Scott, up-complete, and strict.

Let T be a topological space-like non empty reflexive FR-structure. We say that T is order consistent if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let x be an element of T. Then

(i) $\downarrow x = \overline{\{x\}}$, and

(ii) for every eventually-directed net N in T such that $x = \sup N$ and for every neighbourhood V of x holds N is eventually in V.

One can verify that every non empty reflexive topological space-like FR-structure which is trivial is also upper.

Let us mention that there exists a top-lattice which is upper, trivial, up-complete, and strict.

The following propositions are true:

(1) For every upper up-complete non empty top-poset T and for every subset A of T such that A is open holds A is upper.

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(2) For every up-complete non empty top-poset T such that T is upper holds T is order consistent.

(3) Let T be a Scott up-complete non empty reflexive transitive antisymmetric FR-structure and x be an element of T. Then $\downarrow x$ is directly closed and lower.

(4) Let T be a Scott up-complete non empty reflexive transitive antisymmetric FR-structure and S be a subset of T. Then S is closed if and only if S is directly closed and lower.

(5) Let T be a Scott up-complete non empty reflexive transitive antisymmetric FR-structure and x be an element of T. Then $\downarrow x$ is closed.

(6) Let S be an up-complete reflexive antisymmetric non empty relational structure and T be a non empty reflexive relational structure. Suppose the relational structure of $S =$ the relational structure of T. Let A be a subset of S and C be a subset of T. If $A = C$ and A is inaccessible, then C is inaccessible.

(7) For every up-complete non empty reflexive transitive antisymmetric relational structure R holds there exists a topological augmentation of R which is Scott.

(8) Let R be an up-complete non empty poset and T be a topological augmentation of R. If T is Scott, then T is correct.

Let R be an up-complete non empty reflexive transitive antisymmetric relational structure. Observe that every topological augmentation of R which is Scott is also correct.

Let R be an up-complete non empty reflexive transitive antisymmetric relational structure. Note that there exists a topological augmentation of R which is Scott and correct.

The following propositions are true:

(9) Let T be a Scott up-complete non empty reflexive transitive antisymmetric FR-structure and x be an element of T. Then $\overline{\{x\}} = \downarrow x$.

(10) Every up-complete Scott non empty top-poset is order consistent.

(11) Let R be an inf-complete semilattice, Z be a net in R, and D be a subset of R. Suppose $D = \{\bigsqcap_R\{Z(k); k$ ranges over elements of the carrier of Z: $k \geqslant j\} : j$ ranges over elements of the carrier of $Z\}$. Then D is non empty and directed.

(12) Let R be an inf-complete semilattice, S be a subset of R, and a be an element of R. If $a \in S$, then $\bigsqcap_R S \leqslant a$.

(13) For every inf-complete semilattice R and for every monotone reflexive net N in R holds $\liminf N = \sup N$.

(14) Let R be an inf-complete semilattice and S be a subset of R. Then $S \in$ the topology of ConvergenceSpace(the Scott convergence of R) if and

only if S is inaccessible and upper.

(15) Let R be an inf-complete up-complete semilattice and T be a topological augmentation of R. If the topology of $T = \sigma(R)$, then T is Scott.

Let R be an inf-complete up-complete semilattice. One can check that there exists a topological augmentation of R which is strict, Scott, and correct.

One can prove the following two propositions:

(16) Let S be an up-complete inf-complete semilattice and T be a Scott topological augmentation of S. Then $\sigma(S) =$ the topology of T.

(17) Every Scott up-complete non empty reflexive transitive antisymmetric FR-structure is a T_0-space.

Let R be an up-complete non empty reflexive transitive antisymmetric relational structure. Note that every topological augmentation of R is up-complete.

The following propositions are true:

(18) Let R be an up-complete non empty reflexive transitive antisymmetric relational structure, T be a Scott topological augmentation of R, x be an element of T, and A be an upper subset of T. If $x \notin A$, then $-{\downarrow}x$ is a neighbourhood of A.

(19) Let R be an up-complete non empty reflexive transitive antisymmetric FR-structure, T be a Scott topological augmentation of R, and S be an upper subset of T. Then there exists a family F of subsets of T such that $S = \bigcap F$ and for every subset X of T such that $X \in F$ holds X is a neighbourhood of S.

(20) Let T be a Scott up-complete non empty reflexive transitive antisymmetric FR-structure and S be a subset of T. Then S is open if and only if S is upper and property(S).

(21) Let R be an up-complete non empty reflexive transitive antisymmetric FR-structure, S be a non empty directed subset of R, and a be an element of R. If $a \in S$, then $a \leqslant \bigsqcup_R S$.

Let T be an up-complete non empty reflexive transitive antisymmetric FR-structure. One can check that every subset of T which is lower is also property(S).

One can prove the following propositions:

(22) For every finite up-complete non empty poset T holds every subset of T is inaccessible.

(23) Let R be a complete connected lattice, T be a Scott topological augmentation of R, and x be an element of T. Then $-{\downarrow}x$ is open.

(24) Let R be a complete connected lattice, T be a Scott topological augmentation of R, and S be a subset of T. Then S is open if and only if one of the following conditions is satisfied:

(i) $S =$ the carrier of T, or

(ii) $S \in \{-\downarrow x : x$ ranges over elements of $T\}$.

Let R be an up-complete non empty poset. One can check that there exists a correct topological augmentation of R which is order consistent.

Let us observe that there exists a top-lattice which is order consistent and complete.

The following three propositions are true:

(25) Let R be a non empty FR-structure and A be a subset of R. Suppose that for every element x of R holds $\downarrow x = \overline{\{x\}}$. If A is open, then A is upper.

(26) Let R be a non empty FR-structure and A be a subset of R. Suppose that for every element x of R holds $\downarrow x = \overline{\{x\}}$. Let A be a subset of R. If A is closed, then A is lower.

(27) For every up-complete inf-complete lattice T and for every net N in T and for every element i of N holds $\liminf (N \restriction i) = \liminf N$.

Let S be a non empty 1-sorted structure, let R be a non empty relational structure, and let f be a function from the carrier of R into the carrier of S. The functor $R * f$ yielding a strict non empty net structure over S is defined as follows:

(Def. 3) The relational structure of $R * f$ = the relational structure of R and the mapping of $R * f = f$.

Let S be a non empty 1-sorted structure, let R be a non empty transitive relational structure, and let f be a function from the carrier of R into the carrier of S. One can check that $R * f$ is transitive.

Let S be a non empty 1-sorted structure, let R be a non empty directed relational structure, and let f be a function from the carrier of R into the carrier of S. Note that $R * f$ is directed.

Let R be a non empty relational structure and let N be a prenet over R. The functor inf_net N yields a strict prenet over R and is defined by the condition (Def. 4).

(Def. 4) There exists a map f from N into R such that

(i) inf_net $N = N * f$, and

(ii) for every element i of the carrier of N holds $f(i) = \bigsqcap_R \{N(k); k$ ranges over elements of the carrier of N: $k \geqslant i\}$.

Let R be a non empty relational structure and let N be a net in R. One can verify that inf_net N is transitive.

Let R be a non empty relational structure and let N be a net in R. Note that inf_net N is directed.

Let R be an inf-complete non empty reflexive antisymmetric relational structure and let N be a net in R. One can verify that inf_net N is monotone.

Let R be an inf-complete non empty reflexive antisymmetric relational structure and let N be a net in R. One can verify that inf_net N is eventually-directed.

We now state several propositions:

(28) Let R be a non empty relational structure and N be a net in R. Then rng (the mapping of inf_net N) $= \{\bigsqcap_R\{N(i); i$ ranges over elements of the carrier of N: $i \geqslant j\} : j$ ranges over elements of the carrier of $N\}$.

(29) For every up-complete inf-complete lattice R and for every net N in R holds sup inf_net $N = \liminf N$.

(30) For every up-complete inf-complete lattice R and for every net N in R and for every element i of N holds sup inf_net $N = \liminf(N{\restriction}i)$.

(31) Let R be an inf-complete semilattice, N be a net in R, and V be an upper subset of R. If inf_net N is eventually in V, then N is eventually in V.

(32) Let R be an inf-complete semilattice, N be a net in R, and V be a lower subset of R. If N is eventually in V, then inf_net N is eventually in V.

(33) Let R be a topological space-like order consistent up-complete inf-complete non empty top-lattice, N be a net in R, and x be an element of R. If $x \leqslant \liminf N$, then x is a cluster point of N.

(34) Let R be an order consistent up-complete inf-complete topological space-like non empty top-lattice, N be an eventually-directed net in R, and x be an element of R. Then $x \leqslant \liminf N$ if and only if x is a cluster point of N.

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References

[1] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(**5**):719–725, 1991.
[2] Grzegorz Bancerek. Bounds in posets and relational substructures. *Formalized Mathematics*, 6(**1**):81–91, 1997.
[3] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(**1**):93–107, 1997.
[4] Grzegorz Bancerek. Bases and refinements of topologies. *Formalized Mathematics*, 7(**1**):35–43, 1998.
[5] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, Berlin, Heidelberg, New York, 1980.
[6] Artur Korniłowicz. Meet–continuous lattices. *Formalized Mathematics*, 6(**1**):159–167, 1997.
[7] Artur Korniłowicz. On the topological properties of meet-continuous lattices. *Formalized Mathematics*, 6(**2**):269–277, 1997.
[8] Beata Padlewska. Locally connected spaces. *Formalized Mathematics*, 2(**1**):93–96, 1991.
[9] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[10] Andrzej Trybulec. Moore-Smith convergence. *Formalized Mathematics*, 6(**2**):213–225, 1997.
[11] Andrzej Trybulec. Scott topology. *Formalized Mathematics*, 6(**2**):311–319, 1997.
[12] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[13] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[14] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[15] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[16] Mariusz Żynel and Adam Guzowski. T_0 topological spaces. *Formalized Mathematics*, 5(**1**):75–77, 1996.

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