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The Tichonov Theorem

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The terminology and notation used here are introduced in the following articles: [15], [11], [1], [5], [7], [4], [3], [13], [8], [10], [16], [14], [12], [6], [9], and [2].

1. Some Properties of Products

One can prove the following propositions:

(1) For every function F and for all sets i, x_1 and for every subset A_1 of $F(i)$ such that $(\mathrm{proj}(F,i))^{-1}(\{x_1\}) \cap (\mathrm{proj}(F,i))^{-1}(A_1) \neq \emptyset$ holds $x_1 \in A_1$.

(2) For all functions F, f and for all sets i, x_1 such that $x_1 \in F(i)$ and $f \in \prod F$ holds $f +\cdot (i, x_1) \in \prod F$.

(3) For every function F and for every set i such that $i \in \mathrm{dom}\, F$ and $\prod F \neq \emptyset$ holds $\mathrm{rng}\,\mathrm{proj}(F,i) = F(i)$.

(4) For every function F and for every set i such that $i \in \mathrm{dom}\, F$ holds $(\mathrm{proj}(F,i))^{-1}(F(i)) = \prod F$.

(5) For all functions F, f and for all sets i, x_1 such that $x_1 \in F(i)$ and $i \in \mathrm{dom}\, F$ and $f \in \prod F$ holds $f +\cdot (i, x_1) \in (\mathrm{proj}(F,i))^{-1}(\{x_1\})$.

(6) Let F, f be functions, i_1, i_2, x_2 be sets, and A_2 be a subset of $F(i_2)$. Suppose $x_2 \in F(i_1)$ and $i_1 \in \mathrm{dom}\, F$ and $f \in \prod F$. If $i_1 \neq i_2$, then $f \in (\mathrm{proj}(F,i_2))^{-1}(A_2)$ iff $f +\cdot (i_1, x_2) \in (\mathrm{proj}(F,i_2))^{-1}(A_2)$.

(7) Let F be a function, i_1, i_2, x_2 be sets, and A_2 be a subset of $F(i_2)$. Suppose $\prod F \neq \emptyset$ and $x_2 \in F(i_1)$ and $i_1 \in \mathrm{dom}\, F$ and $i_2 \in \mathrm{dom}\, F$ and $A_2 \neq F(i_2)$. Then $(\mathrm{proj}(F,i_1))^{-1}(\{x_2\}) \subseteq (\mathrm{proj}(F,i_2))^{-1}(A_2)$ if and only if $i_1 = i_2$ and $x_2 \in A_2$.

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The scheme *ElProductEx* deals with a non empty set $\mathcal{A}$, a topological space yielding nonempty many sorted set $\mathcal{B}$ indexed by $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

There exists an element f of $\prod \mathcal{B}$ such that for every element i of $\mathcal{A}$ holds $\mathcal{P}[f(i), i]$

provided the parameters have the following property:

- For every element i of $\mathcal{A}$ there exists an element x of $\mathcal{B}(i)$ such that $\mathcal{P}[x, i]$.

One can prove the following propositions:

(8) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, i be an element of I, and f be an element of $\prod J$. Then $(\mathrm{proj}(J, i))(f) = f(i)$.

(9) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, i be an element of I, x_1 be an element of $J(i)$, and A_1 be a subset of $J(i)$. If $(\mathrm{proj}(J, i))^{-1}(\{x_1\}) \cap (\mathrm{proj}(J, i))^{-1}(A_1) \neq \emptyset$, then $x_1 \in A_1$.

(10) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, and i be an element of I. Then $(\mathrm{proj}(J, i))^{-1}(\Omega_{J(i)}) = \Omega_{\prod J}$.

(11) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, i be an element of I, x_1 be an element of $J(i)$, and f be an element of $\prod J$. Then $f +\cdot (i, x_1) \in (\mathrm{proj}(J, i))^{-1}(\{x_1\})$.

(12) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, i_1, i_2 be elements of I, x_2 be an element of $J(i_1)$, and A_2 be a subset of $J(i_2)$. If $A_2 \neq \Omega_{J(i_2)}$, then $(\mathrm{proj}(J, i_1))^{-1}(\{x_2\}) \subseteq (\mathrm{proj}(J, i_2))^{-1}(A_2)$ iff $i_1 = i_2$ and $x_2 \in A_2$.

(13) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, i_1, i_2 be elements of I, x_2 be an element of $J(i_1)$, A_2 be a subset of $J(i_2)$, and f be an element of $\prod J$. If $i_1 \neq i_2$, then $f \in (\mathrm{proj}(J, i_2))^{-1}(A_2)$ iff $f +\cdot (i_1, x_2) \in (\mathrm{proj}(J, i_2))^{-1}(A_2)$.

2. Some Properties of Compact Spaces

One can prove the following three propositions:

(14) Let T be a topological structure and F be a family of subsets of T. Then F is a cover of T if and only if the carrier of $T \subseteq \bigcup F$.

(15) Let T be a non empty topological structure. Then T is compact if and only if for every family F of subsets of T such that F is open and $\Omega_T \subseteq \bigcup F$ there exists a family G of subsets of T such that $G \subseteq F$ and $\Omega_T \subseteq \bigcup G$ and G is finite.

(16) Let T be a non empty topological space and B be a prebasis of T. Then T is compact if and only if for every subset F of B such that $\Omega_T \subseteq \bigcup F$ there exists a finite subset G of F such that $\Omega_T \subseteq \bigcup G$.

3. The Tichonov Theorem

The following propositions are true:

(17) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, and A be a set. Suppose $A \in$ the product prebasis for J. Then there exists an element i of I and there exists a subset A_1 of $J(i)$ such that A_1 is open and $(\mathrm{proj}(J,i))^{-1}(A_1) = A$.

(18) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, i be an element of I, x_1 be an element of $J(i)$, and A be a set. Suppose $A \in$ the product prebasis for J and $(\mathrm{proj}(J,i))^{-1}(\{x_1\}) \subseteq A$. Then $A = \Omega_{\prod J}$ or there exists a subset A_1 of $J(i)$ such that $A_1 \neq \Omega_{J(i)}$ and $x_1 \in A_1$ and A_1 is open and $A = (\mathrm{proj}(J,i))^{-1}(A_1)$.

(19) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, i be an element of I, and F_1 be a non empty family of subsets of $J(i)$. If $\Omega_{J(i)} \subseteq \bigcup F_1$, then $\Omega_{\prod J} \subseteq \bigcup\{(\mathrm{proj}(J,i))^{-1}(A_1) : A_1$ ranges over elements of $F_1\}$.

(20) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, i be an element of I, x_1 be an element of $J(i)$, and G be a subset of the product prebasis for J. Suppose $(\mathrm{proj}(J,i))^{-1}(\{x_1\}) \subseteq \bigcup G$ and for every set A such that $A \in$ the product prebasis for J and $A \in G$ holds $(\mathrm{proj}(J,i))^{-1}(\{x_1\}) \not\subseteq A$. Then $\Omega_{\prod J} \subseteq \bigcup G$.

(21) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, i be an element of I, and F be a subset of the product prebasis for J. Suppose that for every finite subset G of F holds $\Omega_{\prod J} \not\subseteq \bigcup G$. Let x_1 be an element of $J(i)$ and G be a finite subset of F. Suppose $(\mathrm{proj}(J,i))^{-1}(\{x_1\}) \subseteq \bigcup G$. Then there exists a set A such that $A \in$ the product prebasis for J and $A \in G$ and $(\mathrm{proj}(J,i))^{-1}(\{x_1\}) \subseteq A$.

(22) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, i be an element of I, and F be a subset of the product prebasis for J. Suppose that for every finite subset G of F holds $\Omega_{\prod J} \not\subseteq \bigcup G$. Let x_1 be an element of $J(i)$ and G be a finite subset of F. Suppose $(\mathrm{proj}(J,i))^{-1}(\{x_1\}) \subseteq \bigcup G$. Then there exists a subset A_1 of $J(i)$ such that $A_1 \neq \Omega_{J(i)}$ and $x_1 \in A_1$ and $(\mathrm{proj}(J,i))^{-1}(A_1) \in G$ and A_1 is open.

(23) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, i be an element of I, and F be a subset of the product prebasis for J. Suppose for every element i of I holds $J(i)$ is compact and for every finite subset G of F holds $\Omega_{\prod J} \not\subseteq \bigcup G$. Then there exists an element x_1 of $J(i)$ such that for every finite subset G of F holds $(\mathrm{proj}(J,i))^{-1}(\{x_1\}) \not\subseteq \bigcup G$.

(24) Let I be a non empty set and J be a topological space yielding nonempty many sorted set indexed by I. If for every element i of I holds $J(i)$ is compact, then $\prod J$ is compact.

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