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Dynkin's Lemma in Measure Theory

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Summary. This article formalizes the proof of Dynkin's lemma in measure theory. Dynkin's lemma is a useful tool in measure theory and probability theory: it helps frequently to generalize a statement about all elements of a intersection-stable set system to all elements of the sigma-field generated by that system.

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The terminology and notation used in this paper have been introduced in the following articles: [5], [11], [1], [4], [2], [3], [7], [6], [12], [13], [8], [10], and [9].

1. Preliminaries

For simplicity, we adopt the following rules: O_1 denotes a non empty set, f denotes a sequence of subsets of O_1, X, A, B denote subsets of O_1, D denotes a non empty subset of 2^{O_1}, n, m denote natural numbers, F denotes a non empty set, and x, Y denote sets.

Next we state two propositions:

(1) For every sequence f of subsets of O_1 and for every x holds $x \in \operatorname{rng} f$ iff there exists n such that $f(n) = x$.

(2) For every n holds $\mathrm{PSeg}\, n$ is finite.

Let us consider n. One can verify that $\mathrm{PSeg}\, n$ is finite.

Next we state the proposition

(3) For all sets x, y, z such that $x \subseteq y$ holds x misses $z \setminus y$.

Let a, b, c be sets. The functor a, b followed by c is defined as follows:

(Def. 1) a, b followed by $c = (\mathbb{N} \longmapsto c) {+\cdot} [0 \longmapsto a, 1 \longmapsto b]$.

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Let a, b, c be sets. Observe that a, b followed by c is function-like and relation-like.

Let X be a non empty set and let a, b, c be elements of X. Then a, b followed by c is a function from $\mathbb{N}$ into X.

Next we state the proposition

(4) For every non empty set X and for all elements a, b, c of X holds a, b followed by c is a function from $\mathbb{N}$ into X.

Let O_1 be a non empty set and let a, b, c be subsets of O_1. Then a, b followed by c is a sequence of subsets of O_1.

One can prove the following propositions:

(5) For all sets a, b, c holds $(a, b \text{ followed by } c)(0) = a$ and $(a, b \text{ followed by } c)(1) = b$ and for every n such that $n \neq 0$ and $n \neq 1$ holds $(a, b \text{ followed by } c)(n) = c$.

(6) For all subsets a, b of O_1 holds $\bigcup \mathrm{rng}(a, b \text{ followed by } \emptyset) = a \cup b$.

Let O_1 be a non empty set, let f be a sequence of subsets of O_1, and let X be a subset of O_1. The functor seqIntersection(X, f) yields a sequence of subsets of O_1 and is defined by:

(Def. 2) For every n holds $(\mathrm{seqIntersection}(X, f))(n) = X \cap f(n)$.

2. Disjoint-valued Functions and Intersection

Let us consider O_1 and let us consider f. Let us observe that f is disjoint valued if and only if:

(Def. 3) If $n < m$, then $f(n)$ misses $f(m)$.

We now state the proposition

(7) For every non empty set Y and for every x holds $x \subseteq \bigcap Y$ iff for every element y of Y holds $x \subseteq y$.

Let x be a set. We introduce x is intersection stable as a synonym of x is multiplicative.

Let O_1 be a non empty set, let f be a sequence of subsets of O_1, and let n be an element of $\mathbb{N}$. The functor disjointify(f, n) yielding an element of 2^{O_1} is defined by:

(Def. 5)[1] $\mathrm{disjointify}(f, n) = f(n) \setminus \bigcup \mathrm{rng}(f \restriction \mathrm{PSeg}\, n)$.

Let O_1 be a non empty set and let g be a sequence of subsets of O_1. The functor disjointify g yielding a sequence of subsets of O_1 is defined as follows:

(Def. 6) For every n holds $(\mathrm{disjointify}\, g)(n) = \mathrm{disjointify}(g, n)$.

The following propositions are true:

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[1] The definition (Def. 4) has been removed.

(8) For every n holds $(\text{disjointify}\, f)(n) = f(n) \setminus \bigcup \text{rng}(f \restriction \text{PSeg}\, n)$.

(9) For every sequence f of subsets of O_1 holds disjointify f is disjoint valued.

(10) For every sequence f of subsets of O_1 holds $\bigcup \text{rng}\, \text{disjointify}\, f = \bigcup \text{rng}\, f$.

(11) For all subsets x, y of O_1 such that x misses y holds x, y followed by $\emptyset_{(O_1)}$ is disjoint valued.

(12) Let f be a sequence of subsets of O_1. Suppose f is disjoint valued. Let X be a subset of O_1. Then $\text{seqIntersection}(X, f)$ is disjoint valued.

(13) For every sequence f of subsets of O_1 and for every subset X of O_1 holds $X \cap \text{Union}\, f = \text{Union}\, \text{seqIntersection}(X, f)$.

3. Dynkin Systems: Definition and Closure Properties

Let us consider O_1. A subset of 2^{O_1} is called a Dynkin system of O_1 if:

(Def. 7) For every f such that $\text{rng}\, f \subseteq$ it and f is disjoint valued holds $\text{Union}\, f \in$ it and for every X such that $X \in$ it holds $X^{\text{c}} \in$ it and $\emptyset \in$ it.

Let us consider O_1. One can check that every Dynkin system of O_1 is non empty.

The following propositions are true:

(14) 2^{O_1} is a Dynkin system of O_1.

(15) If for every Y such that $Y \in F$ holds Y is a Dynkin system of O_1, then $\bigcap F$ is a Dynkin system of O_1.

(16) If D is a Dynkin system of O_1 and intersection stable, then if $A \in D$ and $B \in D$, then $A \setminus B \in D$.

(17) If D is a Dynkin system of O_1 and intersection stable, then if $A \in D$ and $B \in D$, then $A \cup B \in D$.

(18) Suppose D is a Dynkin system of O_1 and intersection stable. Let x be a finite set. If $x \subseteq D$, then $\bigcup x \in D$.

(19) Suppose D is a Dynkin system of O_1 and intersection stable. Let f be a sequence of subsets of O_1. If $\text{rng}\, f \subseteq D$, then $\text{rng}\, \text{disjointify}\, f \subseteq D$.

(20) Suppose D is a Dynkin system of O_1 and intersection stable. Let f be a sequence of subsets of O_1. If $\text{rng}\, f \subseteq D$, then $\bigcup \text{rng}\, f \in D$.

(21) For every Dynkin system D of O_1 and for all elements x, y of D such that x misses y holds $x \cup y \in D$.

(22) For every Dynkin system D of O_1 and for all elements x, y of D such that $x \subseteq y$ holds $y \setminus x \in D$.

4. Main Steps for Dynkin's Lemma

One can prove the following proposition

(23) If D is a Dynkin system of O_1 and intersection stable, then D is a σ-field of subsets of O_1.

Let O_1 be a non empty set and let E be a subset of 2^{O_1}. The functor $\mathrm{GenDynSys}\, E$ yielding a Dynkin system of O_1 is defined by:

(Def. 8) $E \subseteq \mathrm{GenDynSys}\, E$ and for every Dynkin system D of O_1 such that $E \subseteq D$ holds $\mathrm{GenDynSys}\, E \subseteq D$.

Let O_1 be a non empty set, let G be a set, and let X be a subset of O_1. The functor $\mathrm{DynSys}(X, G)$ yields a subset of 2^{O_1} and is defined as follows:

(Def. 9) For every subset A of O_1 holds $A \in \mathrm{DynSys}(X, G)$ iff $A \cap X \in G$.

Let O_1 be a non empty set, let G be a Dynkin system of O_1, and let X be an element of G. Then $\mathrm{DynSys}(X, G)$ is a Dynkin system of O_1.

Next we state four propositions:

(24) Let E be a subset of 2^{O_1} and X, Y be subsets of O_1. If $X \in E$ and $Y \in \mathrm{GenDynSys}\, E$ and E is intersection stable, then $X \cap Y \in \mathrm{GenDynSys}\, E$.

(25) Let E be a subset of 2^{O_1} and X, Y be subsets of O_1. If $X \in \mathrm{GenDynSys}\, E$ and $Y \in \mathrm{GenDynSys}\, E$ and E is intersection stable, then $X \cap Y \in \mathrm{GenDynSys}\, E$.

(26) For every subset E of 2^{O_1} such that E is intersection stable holds $\mathrm{GenDynSys}\, E$ is intersection stable.

(27) Let E be a subset of 2^{O_1}. Suppose E is intersection stable. Let D be a Dynkin system of O_1. If $E \subseteq D$, then $\sigma(E) \subseteq D$.

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