FORMALIZED MATHEMATICS
Volume 9, Number 3, 2001
University of Białystok

Formal Topological Spaces

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Summary. This article is divided into two parts. In the first part, we prove some useful theorems on finite topological spaces. In the second part, in order to consider a family of neighborhoods to a point in a space, we extend the notion of finite topological space and define a new topological space, which we call formal topological space. We show the relation between formal topological space struct (`FMT_Space_Str`) and the `TopStruct` by giving a function (`NeighSp`). And the following notions are introduced in formal topological spaces: boundary, closure, interior, isolated point, connected point, open set and close set, then some basic facts concerning them are proved. We will discuss the relation between the formal topological space and the finite topological space in future work.

MML Identifier: `FINTOPO2`.

The papers [5], [3], [2], [1], [6], and [4] provide the notation and terminology for this paper.

1. Some Useful Theorems on Finite Topological Spaces

In this paper F_1 denotes a non empty finite topology space and A denotes a subset of the carrier of F_1.

The following propositions are true:

(1) Let F_1 be a non empty finite topology space and A, B be subsets of the carrier of F_1. If $A \subseteq B$, then $A^i \subseteq B^i$.

(2) $A^\delta = A^b \cap (A^i)^c$.

ISSN 1426–2630

(3) $A^\delta = A^b \setminus A^i$.

(4) If A^c is open, then A is closed.

(5) If A^c is closed, then A is open.

Let F_1 be a non empty finite topology space, let x be an element of the carrier of F_1, let y be an element of the carrier of F_1, and let A be a subset of the carrier of F_1. The functor $\mathrm{P}_1(x, y, A)$ yields an element of *Boolean* and is defined by:

(Def. 1) $$\mathrm{P}_1(x, y, A) = \begin{cases} \mathit{true}, & \text{if } y \in U(x) \text{ and } y \in A, \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

Let F_1 be a non empty finite topology space, let x be an element of the carrier of F_1, let y be an element of the carrier of F_1, and let A be a subset of the carrier of F_1. The functor $\mathrm{P}_2(x, y, A)$ yielding an element of *Boolean* is defined as follows:

(Def. 2) $$\mathrm{P}_2(x, y, A) = \begin{cases} \mathit{true}, & \text{if } y \in U(x) \text{ and } y \in A^c, \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

We now state three propositions:

(6) Let x, y be elements of the carrier of F_1 and A be a subset of the carrier of F_1. Then $\mathrm{P}_1(x, y, A) = \mathit{true}$ if and only if $y \in U(x)$ and $y \in A$.

(7) Let x, y be elements of the carrier of F_1 and A be a subset of the carrier of F_1. Then $\mathrm{P}_2(x, y, A) = \mathit{true}$ if and only if $y \in U(x)$ and $y \in A^c$.

(8) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $x \in A^\delta$ if and only if there exist elements y_1, y_2 of the carrier of F_1 such that $\mathrm{P}_1(x, y_1, A) = \mathit{true}$ and $\mathrm{P}_2(x, y_2, A) = \mathit{true}$.

Let F_1 be a non empty finite topology space, let x be an element of the carrier of F_1, and let y be an element of the carrier of F_1. The functor $\mathrm{P}_0(x, y)$ yielding an element of *Boolean* is defined as follows:

(Def. 3) $$\mathrm{P}_0(x, y) = \begin{cases} \mathit{true}, & \text{if } y \in U(x), \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

We now state three propositions:

(9) For all elements x, y of the carrier of F_1 holds $\mathrm{P}_0(x, y) = \mathit{true}$ iff $y \in U(x)$.

(10) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $x \in A^i$ if and only if for every element y of the carrier of F_1 such that $\mathrm{P}_0(x, y) = \mathit{true}$ holds $\mathrm{P}_1(x, y, A) = \mathit{true}$.

(11) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $x \in A^b$ if and only if there exists an element y_1 of the carrier of F_1 such that $\mathrm{P}_1(x, y_1, A) = \mathit{true}$.

Let F_1 be a non empty finite topology space, let x be an element of the carrier of F_1, and let A be a subset of the carrier of F_1. The functor $\mathrm{P}_{\mathrm{A}}(x, A)$ yielding an element of *Boolean* is defined as follows:

(Def. 4) $\mathrm{P_A}(x, A) = \begin{cases} \mathit{true}, & \text{if } x \in A, \\ \mathit{false}, & \text{otherwise.} \end{cases}$

One can prove the following three propositions:

(12) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $\mathrm{P_A}(x, A) = \mathit{true}$ if and only if $x \in A$.

(13) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $x \in A^{\delta_i}$ if and only if the following conditions are satisfied:

(i) there exist elements y_1, y_2 of the carrier of F_1 such that $\mathrm{P_1}(x, y_1, A) = \mathit{true}$ and $\mathrm{P_2}(x, y_2, A) = \mathit{true}$, and

(ii) $\mathrm{P_A}(x, A) = \mathit{true}$.

(14) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $x \in A^{\delta_o}$ if and only if the following conditions are satisfied:

(i) there exist elements y_1, y_2 of the carrier of F_1 such that $\mathrm{P_1}(x, y_1, A) = \mathit{true}$ and $\mathrm{P_2}(x, y_2, A) = \mathit{true}$, and

(ii) $\mathrm{P_A}(x, A) = \mathit{false}$.

Let F_1 be a non empty finite topology space, let x be an element of the carrier of F_1, and let y be an element of the carrier of F_1. The functor $\mathrm{P_e}(x, y)$ yielding an element of *Boolean* is defined by:

(Def. 5) $\mathrm{P_e}(x, y) = \begin{cases} \mathit{true}, & \text{if } x = y, \\ \mathit{false}, & \text{otherwise.} \end{cases}$

The following four propositions are true:

(15) For all elements x, y of the carrier of F_1 holds $\mathrm{P_e}(x, y) = \mathit{true}$ iff $x = y$.

(16) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $x \in A^s$ if and only if the following conditions are satisfied:

(i) $\mathrm{P_A}(x, A) = \mathit{true}$, and

(ii) it is not true that there exists an element y of the carrier of F_1 such that $\mathrm{P_1}(x, y, A) = \mathit{true}$ and $\mathrm{P_e}(x, y) = \mathit{false}$.

(17) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $x \in A^n$ if and only if the following conditions are satisfied:

(i) $\mathrm{P_A}(x, A) = \mathit{true}$, and

(ii) there exists an element y of the carrier of F_1 such that $\mathrm{P_1}(x, y, A) = \mathit{true}$ and $\mathrm{P_e}(x, y) = \mathit{false}$.

(18) Let x be an element of the carrier of F_1 and A be a subset of the carrier of F_1. Then $x \in A^f$ if and only if there exists an element y of the carrier of F_1 such that $\mathrm{P_A}(y, A) = \mathit{true}$ and $\mathrm{P_0}(y, x) = \mathit{true}$.

2. Formal Topological Spaces

We introduce formal topological spaces which are extensions of 1-sorted structure and are systems

⟨ a carrier, a Neighbour-map ⟩,

where the carrier is a set and the Neighbour-map is a function from the carrier into $2^{2^{\text{the carrier}}}$.

Let us observe that there exists a formal topological space which is non empty and strict.

In the sequel T is a non empty topological structure, F_2 is a non empty formal topological space, and x is an element of the carrier of F_2.

Let us consider F_2 and let x be an element of the carrier of F_2. The functor $U_F(x)$ yielding a subset of $2^{\text{the carrier of } F_2}$ is defined as follows:

(Def. 6) $U_F(x) =$ (the Neighbour-map of F_2)(x).

Next we state the proposition

(19) Let F_2 be a non empty formal topological space and x be an element of the carrier of F_2. Then $U_F(x) =$ (the Neighbour-map of F_2)(x).

Let us consider T. The functor NeighSp T yielding a non empty strict formal topological space is defined by the conditions (Def. 7).

(Def. 7)(i) The carrier of NeighSp $T =$ the carrier of T, and

(ii) for every point x of NeighSp T holds $U_F(x) = \{V; V$ ranges over subsets of T: $V \in$ the topology of $T \wedge x \in V\}$.

In the sequel A, B, W, V denote subsets of the carrier of F_2.

Let I_1 be a non empty formal topological space. We say that I_1 is filled if and only if:

(Def. 8) For every element x of the carrier of I_1 and for every subset D of the carrier of I_1 such that $D \in U_F(x)$ holds $x \in D$.

Let us consider F_2 and let us consider A. The functor $A^{F\delta}$ yielding a subset of the carrier of F_2 is defined as follows:

(Def. 9) $A^{F\delta} = \{x : \bigwedge_W (W \in U_F(x) \Rightarrow W \cap A \neq \emptyset \wedge W \cap A^{\rm c} \neq \emptyset)\}$.

The following proposition is true

(20) $x \in A^{F\delta}$ iff for every W such that $W \in U_F(x)$ holds $W \cap A \neq \emptyset$ and $W \cap A^{\rm c} \neq \emptyset$.

Let us consider F_2 and let us consider A. The functor A^{F_b} yielding a subset of the carrier of F_2 is defined as follows:

(Def. 10) $A^{F_b} = \{x : \bigwedge_W (W \in U_F(x) \Rightarrow W \cap A \neq \emptyset)\}$.

One can prove the following proposition

(21) $x \in A^{F_b}$ iff for every W such that $W \in U_F(x)$ holds $W \cap A \neq \emptyset$.

Let us consider F_2 and let us consider A. The functor A^{F_i} yielding a subset of the carrier of F_2 is defined as follows:

(Def. 11) $A^{F_i} = \{x : \bigvee_V \ (V \in U_F(x) \ \wedge \ V \subseteq A)\}$.

Next we state the proposition

(22) $x \in A^{F_i}$ iff there exists V such that $V \in U_F(x)$ and $V \subseteq A$.

Let us consider F_2 and let us consider A. The functor A^{F_s} yields a subset of the carrier of F_2 and is defined by:

(Def. 12) $A^{F_s} = \{x : x \in A \ \wedge \ \bigvee_V \ (V \in U_F(x) \ \wedge \ (V \setminus \{x\}) \cap A = \emptyset)\}$.

One can prove the following proposition

(23) $x \in A^{F_s}$ iff $x \in A$ and there exists V such that $V \in U_F(x)$ and $(V \setminus \{x\}) \cap A = \emptyset$.

Let us consider F_2 and let us consider A. The functor $A^{F_{on}}$ yields a subset of the carrier of F_2 and is defined by:

(Def. 13) $A^{F_{on}} = A \setminus A^{F_s}$.

We now state a number of propositions:

(24) $x \in A^{F_{on}}$ iff $x \in A$ and for every V such that $V \in U_F(x)$ holds $(V \setminus \{x\}) \cap A \neq \emptyset$.

(25) Let F_2 be a non empty formal topological space and A, B be subsets of the carrier of F_2. If $A \subseteq B$, then $A^{F_b} \subseteq B^{F_b}$.

(26) Let F_2 be a non empty formal topological space and A, B be subsets of the carrier of F_2. If $A \subseteq B$, then $A^{F_i} \subseteq B^{F_i}$.

(27) $A^{F_b} \cup B^{F_b} \subseteq A \cup B^{F_b}$.

(28) $A \cap B^{F_b} \subseteq A^{F_b} \cap B^{F_b}$.

(29) $A^{F_i} \cup B^{F_i} \subseteq A \cup B^{F_i}$.

(30) $A \cap B^{F_i} \subseteq A^{F_i} \cap B^{F_i}$.

(31) Let F_2 be a non empty formal topological space. Then the following statements are equivalent

(i) for every element x of the carrier of F_2 and for all subsets V_1, V_2 of the carrier of F_2 such that $V_1 \in U_F(x)$ and $V_2 \in U_F(x)$ there exists a subset W of the carrier of F_2 such that $W \in U_F(x)$ and $W \subseteq V_1 \cap V_2$,

(ii) for all subsets A, B of the carrier of F_2 holds $A \cup B^{F_b} = A^{F_b} \cup B^{F_b}$.

(32) Let F_2 be a non empty formal topological space. Then the following statements are equivalent

(i) for every element x of the carrier of F_2 and for all subsets V_1, V_2 of the carrier of F_2 such that $V_1 \in U_F(x)$ and $V_2 \in U_F(x)$ there exists a subset W of the carrier of F_2 such that $W \in U_F(x)$ and $W \subseteq V_1 \cap V_2$,

(ii) for all subsets A, B of the carrier of F_2 holds $A^{F_i} \cap B^{F_i} = A \cap B^{F_i}$.

(33) Let F_2 be a non empty formal topological space and A, B be subsets of the carrier of F_2. Suppose that for every element x of the carrier of

F_2 and for all subsets V_1, V_2 of the carrier of F_2 such that $V_1 \in U_F(x)$ and $V_2 \in U_F(x)$ there exists a subset W of the carrier of F_2 such that $W \in U_F(x)$ and $W \subseteq V_1 \cap V_2$. Then $A \cup B^{F\delta} \subseteq A^{F\delta} \cup B^{F\delta}$.

(34) Let F_2 be a non empty formal topological space. Suppose F_2 is filled. Suppose that for all subsets A, B of the carrier of F_2 holds $A \cup B^{F\delta} = A^{F\delta} \cup B^{F\delta}$. Let x be an element of the carrier of F_2 and V_1, V_2 be subsets of the carrier of F_2. Suppose $V_1 \in U_F(x)$ and $V_2 \in U_F(x)$. Then there exists a subset W of the carrier of F_2 such that $W \in U_F(x)$ and $W \subseteq V_1 \cap V_2$.

(35) Let x be an element of the carrier of F_2 and A be a subset of the carrier of F_2. Then $x \in A^{F_s}$ if and only if the following conditions are satisfied:

(i) $x \in A$, and

(ii) $x \notin A \setminus \{x\}^{F_b}$.

(36) Let F_2 be a non empty formal topological space. Then F_2 is filled if and only if for every subset A of the carrier of F_2 holds $A \subseteq A^{F_b}$.

(37) Let F_2 be a non empty formal topological space. Then F_2 is filled if and only if for every subset A of the carrier of F_2 holds $A^{F_i} \subseteq A$.

(38) $(A^{\mathrm{c}F_b})^{\mathrm{c}} = A^{F_i}$.

(39) $(A^{\mathrm{c}F_i})^{\mathrm{c}} = A^{F_b}$.

(40) $A^{F\delta} = A^{F_b} \cap A^{\mathrm{c}F_b}$.

(41) $A^{F\delta} = A^{F_b} \cap (A^{F_i})^{\mathrm{c}}$.

(42) $A^{F\delta} = A^{\mathrm{c}F\delta}$.

(43) $A^{F\delta} = A^{F_b} \setminus A^{F_i}$.

Let us consider F_2 and let us consider A. The functor $A^{F\delta_i}$ yields a subset of the carrier of F_2 and is defined by:

(Def. 14) $A^{F\delta_i} = A \cap A^{F\delta}$.

The functor $A^{F\delta_o}$ yields a subset of the carrier of F_2 and is defined by:

(Def. 15) $A^{F\delta_o} = A^{\mathrm{c}} \cap A^{F\delta}$.

The following proposition is true

(44) $A^{F\delta} = A^{F\delta_i} \cup A^{F\delta_o}$.

Let us consider F_2 and let G be a subset of the carrier of F_2. We say that G is open if and only if:

(Def. 16) $G = G^{F_i}$.

We say that G is closed if and only if:

(Def. 17) $G = G^{F_b}$.

Next we state four propositions:

(45) If A^{c} is open, then A is closed.

(46) If A^{c} is closed, then A is open.

(47) Let F_2 be a non empty formal topological space and A, B be subsets of the carrier of F_2. Suppose F_2 is filled. Suppose that for every element x of the carrier of F_2 holds $\{x\} \in U_F(x)$. Then $A \cap B^{F_b} = A^{F_b} \cap B^{F_b}$.

(48) Let F_2 be a non empty formal topological space and A, B be subsets of the carrier of F_2. Suppose F_2 is filled. Suppose that for every element x of the carrier of F_2 holds $\{x\} \in U_F(x)$. Then $A^{F_i} \cup B^{F_i} = A \cup B^{F_i}$.

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Received October 13, 2000