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The Measurability of Extended Real Valued Functions

Noboru Endou
Shinshu University
Nagano

Katsumi Wasaki
Shinshu University
Nagano

Yasunari Shidama
Shinshu University
Nagano

Summary. In this article we prove the measurablility of some extended real valued functions which are $f+g$, $f-g$ and so on. Moreover, we will define the simple function which are defined on the sigma field. It will play an important role for the Lebesgue integral theory.

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The notation and terminology used here are introduced in the following papers: [21], [2], [10], [11], [9], [7], [6], [3], [8], [13], [12], [17], [16], [15], [14], [22], [23], [18], [20], [4], [5], [19], and [1].

1. Finite Valued Function

For simplicity, we adopt the following rules: X is a non empty set, x is an element of X, f, g are partial functions from X to $\overline{\mathbb{R}}$, S is a σ-field of subsets of X, F is a function from $\mathbb{Q}$ into S, p is a rational number, r is a real number, n, m are natural numbers, and A, B are elements of S.

Let us consider X and let us consider f. We say that f is finite if and only if:

(Def. 1) For every x such that $x \in \operatorname{dom} f$ holds $|f(x)| < +\infty$.

Next we state three propositions:

(1) $f = 1\, f$.

(2) For all f, g, A such that f is finite or g is finite holds $\operatorname{dom}(f + g) = \operatorname{dom} f \cap \operatorname{dom} g$ and $\operatorname{dom}(f - g) = \operatorname{dom} f \cap \operatorname{dom} g$.

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(3) Let given f, g, F, r, A. Suppose f is finite and g is finite and for every p holds $F(p) = A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(p)) \cap (A \cap \text{LE-dom}(g, \overline{\mathbb{R}}(r-p)))$. Then $A \cap \text{LE-dom}(f+g, \overline{\mathbb{R}}(r)) = \bigcup \operatorname{rng} F$.

2. Measurability of $f+g$ and $f-g$

The following propositions are true:

(4) There exists a function F from $\mathbb{N}$ into $\mathbb{Q}$ such that F is one-to-one and $\operatorname{dom} F = \mathbb{N}$ and $\operatorname{rng} F = \mathbb{Q}$.

(5) Let X, Y, Z be non empty sets and F be a function from X into Z. If $X \approx Y$, then there exists a function G from Y into Z such that $\operatorname{rng} F = \operatorname{rng} G$.

(6) Let given S, f, g, A. Suppose f is measurable on A and g is measurable on A. Then there exists a function F from $\mathbb{Q}$ into S such that for every rational number p holds $F(p) = A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(p)) \cap (A \cap \text{LE-dom}(g, \overline{\mathbb{R}}(r-p)))$.

(7) Let given f, g, A. Suppose f is finite and g is finite and f is measurable on A and g is measurable on A. Then $f+g$ is measurable on A.

(8) For all sets E, F, G and for every partial function f from E to F holds $f^{-1}(G) \subseteq E$.

(9) For every non empty set C and for all partial functions f_1, f_2 from C to $\overline{\mathbb{R}}$ holds $f_1 - f_2 = f_1 + -f_2$.

(10) For every real number r holds $\overline{\mathbb{R}}(-r) = -\overline{\mathbb{R}}(r)$.

(11) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $-f = (-1)\, f$.

(12) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and r be a real number. If f is finite, then $r\, f$ is finite.

(13) Let given f, g, A. Suppose f is finite and g is finite and f is measurable on A and g is measurable on A and $A \subseteq \operatorname{dom} g$. Then $f-g$ is measurable on A.

3. Definitions of Extended Real Valued Functions $\max_+(f)$ and $\max_-(f)$ and their Basic Properties

Let C be a non empty set and let f be a partial function from C to $\overline{\mathbb{R}}$. The functor $\max_+(f)$ yields a partial function from C to $\overline{\mathbb{R}}$ and is defined as follows:

(Def. 2) $\operatorname{dom} \max_+(f) = \operatorname{dom} f$ and for every element x of C such that $x \in \operatorname{dom} \max_+(f)$ holds $(\max_+(f))(x) = \max(f(x), 0_{\overline{\mathbb{R}}})$.

The functor $\max_-(f)$ yielding a partial function from C to $\overline{\mathbb{R}}$ is defined by:

(Def. 3) $\operatorname{dom}\max_-(f) = \operatorname{dom} f$ and for every element x of C such that $x \in \operatorname{dom}\max_-(f)$ holds $(\max_-(f))(x) = \max(-f(x), 0_{\overline{\mathbb{R}}})$.

The following propositions are true:

(14) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$, then $0_{\overline{\mathbb{R}}} \leqslant (\max_+(f))(x)$.

(15) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$, then $0_{\overline{\mathbb{R}}} \leqslant (\max_-(f))(x)$.

(16) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $\max_-(f) = \max_+(-f)$.

(17) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $0_{\overline{\mathbb{R}}} < (\max_+(f))(x)$, then $(\max_-(f))(x) = 0_{\overline{\mathbb{R}}}$.

(18) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $0_{\overline{\mathbb{R}}} < (\max_-(f))(x)$, then $(\max_+(f))(x) = 0_{\overline{\mathbb{R}}}$.

(19) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $\operatorname{dom} f = \operatorname{dom}(\max_+(f) - \max_-(f))$ and $\operatorname{dom} f = \operatorname{dom}(\max_+(f) + \max_-(f))$.

(20) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$, then $(\max_+(f))(x) = f(x)$ or $(\max_+(f))(x) = 0_{\overline{\mathbb{R}}}$ but $(\max_-(f))(x) = -f(x)$ or $(\max_-(f))(x) = 0_{\overline{\mathbb{R}}}$.

(21) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $(\max_+(f))(x) = f(x)$, then $(\max_-(f))(x) = 0_{\overline{\mathbb{R}}}$.

(22) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $(\max_+(f))(x) = 0_{\overline{\mathbb{R}}}$, then $(\max_-(f))(x) = -f(x)$.

(23) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $(\max_-(f))(x) = -f(x)$, then $(\max_+(f))(x) = 0_{\overline{\mathbb{R}}}$.

(24) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $(\max_-(f))(x) = 0_{\overline{\mathbb{R}}}$, then $(\max_+(f))(x) = f(x)$.

(25) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $f = \max_+(f) - \max_-(f)$.

(26) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $|f| = \max_+(f) + \max_-(f)$.

4. Measurability of $\max_+(f)$, $\max_-(f)$ and $|f|$

Next we state three propositions:

(27) If f is measurable on A, then $\max_+(f)$ is measurable on A.

(28) If f is measurable on A and $A \subseteq \operatorname{dom} f$, then $\max_-(f)$ is measurable on A.

(29) For all f, A such that f is measurable on A and $A \subseteq \operatorname{dom} f$ holds $|f|$ is measurable on A.

5. Definition and Measurability of Characteristic Function

One can prove the following proposition

(30) For all sets A, X holds $\operatorname{rng}(\chi_{A,X}) \subseteq \{0_{\overline{\mathbb{R}}}, \overline{1}\}$.

Let A, X be sets. Then $\chi_{A,X}$ is a partial function from X to $\overline{\mathbb{R}}$.

Next we state two propositions:

(31) $\chi_{A,X}$ is finite.

(32) $\chi_{A,X}$ is measurable on B.

6. Definition and Measurability of Simple Function

Let X be a set and let S be a σ-field of subsets of X. One can check that there exists a finite sequence of elements of S which is disjoint valued.

Let X be a set and let S be a σ-field of subsets of X. A finite sequence of separated subsets of S is a disjoint valued finite sequence of elements of S.

The following propositions are true:

(33) Suppose F is a finite sequence of separated subsets of S. Then there exists a sequence G of separated subsets of S such that $\bigcup \operatorname{rng} F = \bigcup \operatorname{rng} G$ and for every n such that $n \in \operatorname{dom} F$ holds $F(n) = G(n)$ and for every m such that $m \notin \operatorname{dom} F$ holds $G(m) = \emptyset$.

(34) If F is a finite sequence of separated subsets of S, then $\bigcup \operatorname{rng} F \in S$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let f be a partial function from X to $\overline{\mathbb{R}}$. We say that f is simple function in S if and only if the conditions (Def. 5) are satisfied.

(Def. 5)[1] (i) f is finite, and

(ii) there exists a finite sequence F of separated subsets of S such that $\operatorname{dom} f = \bigcup \operatorname{rng} F$ and for every natural number n and for all elements x, y of X such that $n \in \operatorname{dom} F$ and $x \in F(n)$ and $y \in F(n)$ holds $f(x) = f(y)$.

[1] The definition (Def. 4) has been removed.

One can prove the following propositions:

(35) If f is finite, then $\operatorname{rng} f$ is a subset of $\mathbb{R}$.

(36) Suppose F is a finite sequence of separated subsets of S. Let given n. Then $F \restriction \operatorname{Seg} n$ is a finite sequence of separated subsets of S.

(37) If f is simple function in S, then f is measurable on A.

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