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Again on the Order on a Special Polygon[1]

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The terminology and notation used in this paper have been introduced in the following articles: [6], [2], [14], [4], [12], [3], [11], [1], [5], [8], [16], [10], [9], [13], [15], and [7].

1. Preliminaries

For simplicity, we use the following convention: D denotes a non empty set, f denotes a finite sequence of elements of D, g denotes a circular finite sequence of elements of D, and p, p_1, p_2, p_3, q denote elements of D.

We now state several propositions:

(1) If $q \in \operatorname{rng}(f \restriction p \looparrowleft f)$, then $q \looparrowleft f \leqslant p \looparrowleft f$.

(2) If $p \in \operatorname{rng} f$ and $q \in \operatorname{rng} f$ and $p \looparrowleft f \leqslant q \looparrowleft f$, then $q \looparrowleft (f :- p) = (q \looparrowleft f - p \looparrowleft f) + 1$.

(3) If $p \in \operatorname{rng} f$ and $q \in \operatorname{rng} f$ and $p \looparrowleft f < q \looparrowleft f$, then $p \looparrowleft (f -: q) = p \looparrowleft f$.

(4) If $p \in \operatorname{rng} f$ and $q \in \operatorname{rng} f$ and $p \looparrowleft f \leqslant q \looparrowleft f$, then $q \looparrowleft (f^{p}_{\circlearrowleft}) = (q \looparrowleft f - p \looparrowleft f) + 1$.

(5) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f$ and $p_3 \in \operatorname{rng} f$ and $p_1 \looparrowleft f \leqslant p_2 \looparrowleft f$ and $p_2 \looparrowleft f < p_3 \looparrowleft f$, then $p_2 \looparrowleft (f^{p_1}_{\circlearrowleft}) < p_3 \looparrowleft (f^{p_1}_{\circlearrowleft})$.

(6) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f$ and $p_3 \in \operatorname{rng} f$ and $p_1 \looparrowleft f < p_2 \looparrowleft f$ and $p_2 \looparrowleft f \leqslant p_3 \looparrowleft f$, then $p_2 \looparrowleft (f^{p_1}_{\circlearrowleft}) \leqslant p_3 \looparrowleft (f^{p_1}_{\circlearrowleft})$.

(7) If $p \in \operatorname{rng} g$ and $\operatorname{len} g > 1$, then $p \looparrowleft g < \operatorname{len} g$.

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2. Ordering of Special Points on a Standard Special Sequence

We adopt the following rules: f denotes a non constant standard special circular sequence and p, p_1, p_2, p_3, q denote points of $\mathcal{E}^2_{\rm T}$.

The following propositions are true:

(8) $f_{\lfloor 1}$ is one-to-one.

(9) If $1 < q \hookleftarrow f$ and $q \in \operatorname{rng} f$, then $(\pi_1 f) \hookleftarrow (f^{q}_{\circlearrowleft}) = (\operatorname{len} f + 1) - q \hookleftarrow f$.

(10) If $p \in \operatorname{rng} f$ and $q \in \operatorname{rng} f$ and $p \hookleftarrow f < q \hookleftarrow f$, then $p \hookleftarrow (f^{q}_{\circlearrowleft}) = (\operatorname{len} f + p \hookleftarrow f) - q \hookleftarrow f$.

(11) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f$ and $p_3 \in \operatorname{rng} f$ and $p_1 \hookleftarrow f < p_2 \hookleftarrow f$ and $p_2 \hookleftarrow f < p_3 \hookleftarrow f$, then $p_3 \hookleftarrow (f^{p_2}_{\circlearrowleft}) < p_1 \hookleftarrow (f^{p_2}_{\circlearrowleft})$.

(12) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f$ and $p_3 \in \operatorname{rng} f$ and $p_1 \hookleftarrow f < p_2 \hookleftarrow f$ and $p_2 \hookleftarrow f < p_3 \hookleftarrow f$, then $p_1 \hookleftarrow (f^{p_3}_{\circlearrowleft}) < p_2 \hookleftarrow (f^{p_3}_{\circlearrowleft})$.

(13) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f$ and $p_3 \in \operatorname{rng} f$ and $p_1 \hookleftarrow f \leqslant p_2 \hookleftarrow f$ and $p_2 \hookleftarrow f < p_3 \hookleftarrow f$, then $p_1 \hookleftarrow (f^{p_3}_{\circlearrowleft}) \leqslant p_2 \hookleftarrow (f^{p_3}_{\circlearrowleft})$.

(14) $(\text{S-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < \operatorname{len} f$.

(15) $(\text{S-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < \operatorname{len} f$.

(16) $(\text{E-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < \operatorname{len} f$.

(17) $(\text{E-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < \operatorname{len} f$.

(18) $(\text{N-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < \operatorname{len} f$.

(19) $(\text{N-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < \operatorname{len} f$.

(20) $(\text{W-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < \operatorname{len} f$.

(21) $(\text{W-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < \operatorname{len} f$.

3. Ordering of Special Points on a Clockwise Oriented Sequence

In the sequel z is a clockwise oriented non constant standard special circular sequence.

Next we state a number of propositions:

(22) If $\pi_1 f = \text{W-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{W-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(23) If $\pi_1 f = \text{W-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f > 1$.

(24) If $\pi_1 z = \text{W-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{W-max}\,\widetilde{\mathcal{L}}(z) \neq \text{N-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{N-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(25) If $\pi_1 z = \text{W-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{N-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(26) If $\pi_1 z = \text{W-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{N-max}\,\widetilde{\mathcal{L}}(z) \neq \text{E-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{E-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(27) If $\pi_1 z = \text{W-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{E-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(28) If $\pi_1 z = \text{W-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{E-min}\,\widetilde{\mathcal{L}}(z) \neq \text{S-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{S-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(29) If $\pi_1 z = \text{W-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{S-min}\,\widetilde{\mathcal{L}}(z) \neq \text{W-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{S-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(30) If $\pi_1 f = \text{S-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{S-max}\,\widetilde{\mathcal{L}}(f)) \looparrowleft f < (\text{S-min}\,\widetilde{\mathcal{L}}(f)) \looparrowleft f$.

(31) If $\pi_1 f = \text{S-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{S-min}\,\widetilde{\mathcal{L}}(f)) \looparrowleft f > 1$.

(32) If $\pi_1 z = \text{S-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{S-min}\,\widetilde{\mathcal{L}}(z) \neq \text{W-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{W-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(33) If $\pi_1 z = \text{S-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{W-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(34) If $\pi_1 z = \text{S-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{W-max}\,\widetilde{\mathcal{L}}(z) \neq \text{N-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{N-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(35) If $\pi_1 z = \text{S-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{N-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(36) If $\pi_1 z = \text{S-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{N-max}\,\widetilde{\mathcal{L}}(z) \neq \text{E-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{E-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(37) If $\pi_1 z = \text{S-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{E-min}\,\widetilde{\mathcal{L}}(z) \neq \text{S-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{E-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(38) If $\pi_1 f = \text{E-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{E-max}\,\widetilde{\mathcal{L}}(f)) \looparrowleft f < (\text{E-min}\,\widetilde{\mathcal{L}}(f)) \looparrowleft f$.

(39) If $\pi_1 f = \text{E-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(f)) \looparrowleft f > 1$.

(40) If $\pi_1 z = \text{E-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{S-max}\,\widetilde{\mathcal{L}}(z) \neq \text{E-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{S-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(41) If $\pi_1 z = \text{E-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{S-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(42) If $\pi_1 z = \text{E-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{S-min}\,\widetilde{\mathcal{L}}(z) \neq \text{W-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{W-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(43) If $\pi_1 z = \text{E-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{W-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(44) If $\pi_1 z = \text{E-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{W-max}\,\widetilde{\mathcal{L}}(z) \neq \text{N-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{N-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(45) If $\pi_1 z = \text{E-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{N-max}\,\widetilde{\mathcal{L}}(z) \neq \text{E-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{N-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(46) If $\pi_1 f = \text{N-max}\,\widetilde{\mathcal{L}}(f)$ and $\text{N-max}\,\widetilde{\mathcal{L}}(f) \neq \text{E-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{N-max}\,\widetilde{\mathcal{L}}(f)) \looparrowleft f < (\text{E-max}\,\widetilde{\mathcal{L}}(f)) \looparrowleft f$.

(47) If $\pi_1 z = \text{N-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{E-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(48) If $\pi_1 z = \text{N-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{E-min}\,\widetilde{\mathcal{L}}(z) \neq \text{S-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{S-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(49) If $\pi_1 z = \text{N-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{S-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(50) If $\pi_1 z = \text{N-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{S-min}\,\widetilde{\mathcal{L}}(z) \neq \text{W-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{W-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(51) If $\pi_1 z = \text{N-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z < (\text{W-max}\,\widetilde{\mathcal{L}}(z)) \looparrowleft z$.

(52) If $\pi_1 z = \text{N-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{N-min}\,\widetilde{\mathcal{L}}(z) \neq \text{W-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{N-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(53) If $\pi_1 f = \text{E-min}\,\widetilde{\mathcal{L}}(f)$ and $\text{E-min}\,\widetilde{\mathcal{L}}(f) \neq \text{S-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{S-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(54) If $\pi_1 z = \text{E-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{S-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(55) If $\pi_1 z = \text{E-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{S-min}\,\widetilde{\mathcal{L}}(z) \neq \text{W-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{W-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(56) If $\pi_1 z = \text{E-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{W-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(57) If $\pi_1 z = \text{E-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{W-max}\,\widetilde{\mathcal{L}}(z) \neq \text{N-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{N-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(58) If $\pi_1 z = \text{E-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{N-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(59) If $\pi_1 z = \text{E-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{E-max}\,\widetilde{\mathcal{L}}(z) \neq \text{N-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{E-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(60) If $\pi_1 f = \text{S-min}\,\widetilde{\mathcal{L}}(f)$ and $\text{S-min}\,\widetilde{\mathcal{L}}(f) \neq \text{W-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{S-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{W-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(61) If $\pi_1 z = \text{S-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{W-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(62) If $\pi_1 z = \text{S-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{W-max}\,\widetilde{\mathcal{L}}(z) \neq \text{N-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{N-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(63) If $\pi_1 z = \text{S-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{N-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(64) If $\pi_1 z = \text{S-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{N-max}\,\widetilde{\mathcal{L}}(z) \neq \text{E-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{E-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(65) If $\pi_1 z = \text{S-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{E-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(66) If $\pi_1 z = \text{S-min}\,\widetilde{\mathcal{L}}(z)$ and $\text{S-max}\,\widetilde{\mathcal{L}}(z) \neq \text{E-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{S-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(67) If $\pi_1 f = \text{W-max}\,\widetilde{\mathcal{L}}(f)$ and $\text{W-max}\,\widetilde{\mathcal{L}}(f) \neq \text{N-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{N-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(68) If $\pi_1 z = \text{W-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{N-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(69) If $\pi_1 z = \text{W-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{N-max}\,\widetilde{\mathcal{L}}(z) \neq \text{E-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{N-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{E-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(70) If $\pi_1 z = \text{W-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{E-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(71) If $\pi_1 z = \text{W-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{E-min}\,\widetilde{\mathcal{L}}(z) \neq \text{S-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{S-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(72) If $\pi_1 z = \text{W-max}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-max}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{S-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

(73) If $\pi_1 z = \text{W-max}\,\widetilde{\mathcal{L}}(z)$ and $\text{W-min}\,\widetilde{\mathcal{L}}(z) \neq \text{S-min}\,\widetilde{\mathcal{L}}(z)$, then $(\text{S-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z < (\text{W-min}\,\widetilde{\mathcal{L}}(z)) \hookleftarrow z$.

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