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The Urysohn Lemma

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Summary. This article is the third part of a paper proving the fundamental Urysohn Theorem concerning the existence of a real valued continuous function on a normal topological space. The paper is divided into two parts. In the first part, we describe the construction of the function solving thesis of the Urysohn Lemma. The second part contains the proof of the Urysohn Lemma in normal space and the proof of the same theorem for compact space.

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The notation and terminology used here have been introduced in the following papers: [15], [10], [7], [8], [4], [1], [9], [6], [12], [16], [17], [13], [14], [2], [3], [11], and [5].

Let D be a non empty subset of $\mathbb{R}$. One can check that every element of D is real.

One can prove the following proposition

(1) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let n be a natural number. Then there exists a function G from $\mathrm{dyadic}(n)$ into $2^{\text{the carrier of } T}$ such that for all elements r_1, r_2 of $\mathrm{dyadic}(n)$ if $r_1 < r_2$, then $G(r_1)$ is open and $G(r_2)$ is open and $\overline{G(r_1)} \subseteq G(r_2)$ and $A \subseteq G(0)$ and $B = \Omega_T \setminus G(1)$.

Let T be a non empty topological space, let A, B be subsets of T, and let n be a natural number. Let us assume that T is a T_4 space and $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. A function from $\mathrm{dyadic}(n)$ into $2^{\text{the carrier of } T}$ is said to be a drizzle of A, B, n if it satisfies the condition (Def. 1).

(Def. 1) Let r_1, r_2 be elements of $\mathrm{dyadic}(n)$. Suppose $r_1 < r_2$. Then $\mathrm{it}(r_1)$ is open and $\mathrm{it}(r_2)$ is open and $\overline{\mathrm{it}(r_1)} \subseteq \mathrm{it}(r_2)$ and $A \subseteq \mathrm{it}(0)$ and $B = \Omega_T \setminus \mathrm{it}(1)$.

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One can prove the following propositions:

(2) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let n be a natural number and D be a drizzle of A, B, n. Then $A \subseteq D(0)$ and $B = \Omega_T \setminus D(1)$.

(3) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let n be a natural number and G be a drizzle of A, B, n. Then there exists a drizzle F of A, B, $n+1$ such that for every element r of dyadic$(n+1)$ if $r \in$ dyadic(n), then $F(r) = G(r)$.

Let A, B be non empty sets, let F be a function from $\mathbb{N}$ into $A \dot{\rightarrow} B$, and let n be a natural number. Then $F(n)$ is a partial function from A to B.

Next we state the proposition

(4) Let T be a non empty topological space, A, B be subsets of T, and n be a natural number. Then every drizzle of A, B, n is an element of DYADIC $\dot{\rightarrow} 2^{\text{the carrier of } T}$.

Let A, B be non empty sets, let F be a function from $\mathbb{N}$ into $A \dot{\rightarrow} B$, and let n be a natural number. Then $F(n)$ is an element of $A \dot{\rightarrow} B$.

One can prove the following proposition

(5) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Then there exists a sequence F of partial functions from DYADIC into $2^{\text{the carrier of } T}$ such that for every natural number n holds $F(n)$ is a drizzle of A, B, n and for every element r of dom $F(n)$ holds $F(n)(r) = F(n+1)(r)$.

Let T be a non empty topological space and let A, B be subsets of T. Let us assume that T is a T_4 space and $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. A sequence of partial functions from DYADIC into $2^{\text{the carrier of } T}$ is said to be a rain of A, B if it satisfies the condition (Def. 2).

(Def. 2) Let n be a natural number. Then it(n) is a drizzle of A, B, n and for every element r of dom it(n) holds it$(n)(r) =$ it$(n+1)(r)$.

Let x be a real number. Let us assume that $x \in$ DYADIC. The functor InfDyadic x yields a natural number and is defined by:

(Def. 3) $x \in$ dyadic(0) iff InfDyadic $x = 0$ and for every natural number n such that $x \in$ dyadic$(n+1)$ and $x \notin$ dyadic(n) holds InfDyadic $x = n+1$.

The following propositions are true:

(6) For every real number x such that $x \in$ DYADIC holds $x \in$ dyadic(InfDyadic x).

(7) For every real number x such that $x \in$ DYADIC and for every natural number n such that InfDyadic $x \leqslant n$ holds $x \in$ dyadic(n).

(8) For every real number x such that $x \in$ DYADIC and for every natural number n such that $x \in \mathrm{dyadic}(n)$ holds InfDyadic $x \leqslant n$.

(9) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B and x be a real number. If $x \in$ DYADIC, then for every natural number n holds $G(\mathrm{InfDyadic}\, x)(x) = G(\mathrm{InfDyadic}\, x + n)(x)$.

(10) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B and x be a real number. Suppose $x \in$ DYADIC . Then there exists an element y of $2^{\text{the carrier of } T}$ such that for every natural number n if $x \in \mathrm{dyadic}(n)$, then $y = G(n)(x)$.

(11) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B. Then there exists a function F from DOM into $2^{\text{the carrier of } T}$ such that for every real number x holds

(i) if $x \in \mathbb{R}_{<0}$, then $F(x) = \emptyset$,

(ii) if $x \in \mathbb{R}_{>1}$, then $F(x) =$ the carrier of T, and

(iii) if $x \in$ DYADIC, then for every natural number n such that $x \in \mathrm{dyadic}(n)$ holds $F(x) = G(n)(x)$.

Let T be a non empty topological space and let A, B be subsets of T. Let us assume that T is a T_4 space and $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let R be a rain of A, B. The functor Tempest R yielding a function from DOM into $2^{\text{the carrier of } T}$ is defined by the condition (Def. 4).

(Def. 4) Let x be a real number such that $x \in$ DOM . Then

(i) if $x \in \mathbb{R}_{<0}$, then $(\mathrm{Tempest}\, R)(x) = \emptyset$,

(ii) if $x \in \mathbb{R}_{>1}$, then $(\mathrm{Tempest}\, R)(x) =$ the carrier of T, and

(iii) if $x \in$ DYADIC, then for every natural number n such that $x \in \mathrm{dyadic}(n)$ holds $(\mathrm{Tempest}\, R)(x) = R(n)(x)$.

Let X be a non empty set, let T be a topological space, let F be a function from X into $2^{\text{the carrier of } T}$, and let x be an element of X. Then $F(x)$ is a subset of T.

One can prove the following three propositions:

(12) Let T be a non empty topological space and A, B be subsets of T. Suppose T is a T_4 space and $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B and r be a real number. If $r \in$ DOM, then for every subset C of T such that $C = (\mathrm{Tempest}\, G)(r)$ holds C is open.

(13) Let T be a non empty topological space and A, B be subsets of T. Suppose T is a T_4 space and $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B and r_1, r_2 be real numbers. Suppose

$r_1 \in$ DOM and $r_2 \in$ DOM and $r_1 < r_2$. Let C be a subset of T. If $C = (\text{Tempest}\, G)(r_1)$, then $\overline{C} \subseteq (\text{Tempest}\, G)(r_2)$.

(14) Let T be a non empty topological space, A, B be subsets of T, G be a rain of A, B, and p be a point of T. Then there exists a subset R of $\overline{\mathbb{R}}$ such that for every set x holds $x \in R$ if and only if the following conditions are satisfied:
  (i) $x \in$ DYADIC, and
  (ii) for every real number s such that $s = x$ holds $p \notin (\text{Tempest}\, G)(s)$.

Let T be a non empty topological space, let A, B be subsets of T, let R be a rain of A, B, and let p be a point of T. The functor $\text{Rainbow}(p, R)$ yielding a subset of $\overline{\mathbb{R}}$ is defined by:

(Def. 5) For every set x holds $x \in \text{Rainbow}(p, R)$ iff $x \in$ DYADIC and for every real number s such that $s = x$ holds $p \notin (\text{Tempest}\, R)(s)$.

Let T, S be non empty topological spaces, let F be a function from the carrier of T into the carrier of S, and let p be a point of T. Then $F(p)$ is a point of S.

One can prove the following propositions:

(15) Let T be a non empty topological space, A, B be subsets of T, G be a rain of A, B, and p be a point of T. Then $\text{Rainbow}(p, G) \subseteq$ DYADIC.

(16) Let T be a non empty topological space, A, B be subsets of T, and R be a rain of A, B. Then there exists a map F from T into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of T holds
if $\text{Rainbow}(p, R) = \emptyset$, then $F(p) = 0$ and for every non empty subset S of $\overline{\mathbb{R}}$ such that $S = \text{Rainbow}(p, R)$ holds $F(p) = \sup S$.

Let T be a non empty topological space, let A, B be subsets of T, and let R be a rain of A, B. The functor $\text{Thunder}\, R$ yielding a map from T into $\mathbb{R}^{\mathbf{1}}$ is defined by the condition (Def. 6).

(Def. 6) Let p be a point of T. Then if $\text{Rainbow}(p, R) = \emptyset$, then $(\text{Thunder}\, R)(p) = 0$ and for every non empty subset S of $\overline{\mathbb{R}}$ such that $S = \text{Rainbow}(p, R)$ holds $(\text{Thunder}\, R)(p) = \sup S$.

Let T be a non empty topological space, let F be a map from T into $\mathbb{R}^{\mathbf{1}}$, and let p be a point of T. Then $F(p)$ is a real number.

One can prove the following propositions:

(17) Let T be a non empty topological space, A, B be subsets of T, G be a rain of A, B, p be a point of T, and S be a non empty subset of $\overline{\mathbb{R}}$. Suppose $S = \text{Rainbow}(p, G)$. Let ℓ_1 be an extended real number. If $\ell_1 = 1$, then $0_{\overline{\mathbb{R}}} \leqslant \sup S$ and $\sup S \leqslant \ell_1$.

(18) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B, r be an element of DOM, and p be a

point of T. If $(\mathrm{Thunder}\, G)(p) < r$, then $p \in (\mathrm{Tempest}\, G)(r)$.

(19) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B and r be a real number. Suppose $r \in \mathrm{DYADIC} \cup \mathbb{R}_{>1}$ and $0 < r$. Let p be a point of T. If $p \in (\mathrm{Tempest}\, G)(r)$, then $(\mathrm{Thunder}\, G)(p) \leqslant r$.

(20) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B, n be a natural number, and r_1 be an element of DOM. If $0 < r_1$, then for every point p of T such that $r_1 < (\mathrm{Thunder}\, G)(p)$ holds $p \notin (\mathrm{Tempest}\, G)(r_1)$.

(21) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let G be a rain of A, B. Then

(i) $\mathrm{Thunder}\, G$ is continuous, and

(ii) for every point x of T holds $0 \leqslant (\mathrm{Thunder}\, G)(x)$ and $(\mathrm{Thunder}\, G)(x) \leqslant 1$ and if $x \in A$, then $(\mathrm{Thunder}\, G)(x) = 0$ and if $x \in B$, then $(\mathrm{Thunder}\, G)(x) = 1$.

(22) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Then there exists a map F from T into $\mathbb{R}^{\mathbf{1}}$ such that

(i) F is continuous, and

(ii) for every point x of T holds $0 \leqslant F(x)$ and $F(x) \leqslant 1$ and if $x \in A$, then $F(x) = 0$ and if $x \in B$, then $F(x) = 1$.

(23) Let T be a non empty topological space. Suppose T is a T_4 space. Let A, B be subsets of T. Suppose A is closed and B is closed and $A \cap B = \emptyset$. Then there exists a map F from T into $\mathbb{R}^{\mathbf{1}}$ such that

(i) F is continuous, and

(ii) for every point x of T holds $0 \leqslant F(x)$ and $F(x) \leqslant 1$ and if $x \in A$, then $F(x) = 0$ and if $x \in B$, then $F(x) = 1$.

(24) Let T be a non empty topological space. Suppose T is a T_2 space and compact. Let A, B be subsets of T. Suppose A is closed and B is closed and $A \cap B = \emptyset$. Then there exists a map F from T into $\mathbb{R}^{\mathbf{1}}$ such that

(i) F is continuous, and

(ii) for every point x of T holds $0 \leqslant F(x)$ and $F(x) \leqslant 1$ and if $x \in A$, then $F(x) = 0$ and if $x \in B$, then $F(x) = 1$.

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