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On Outside Fashoda Meet Theorem

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Summary. We have proven the "Fashoda Meet Theorem" in [12]. Here we prove the outside version of it. It says that if Britain and France intended to set the courses for ships to the opposite side of Africa, they must also meet.

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The articles [19], [8], [1], [2], [3], [4], [12], [13], [11], [5], [14], [7], [10], [20], [17], [18], [16], [9], [15], and [6] provide the terminology and notation for this paper.

One can prove the following propositions:

(1) For all real numbers a, b such that $a \neq 0$ and $b \neq 0$ holds $\frac{a}{b} \cdot \frac{b}{a} = 1$.

(2) For every real number a such that $1 \leqslant a$ holds $a \leqslant a^{\mathbf{2}}$.

(3) For every real number a such that $-1 \geqslant a$ holds $-a \leqslant a^{\mathbf{2}}$.

(4) For every real number a such that $-1 > a$ holds $-a < a^{\mathbf{2}}$.

(5) For all real numbers a, b such that $b^{\mathbf{2}} \leqslant a^{\mathbf{2}}$ and $a \geqslant 0$ holds $-a \leqslant b$ and $b \leqslant a$.

(6) For all real numbers a, b such that $b^{\mathbf{2}} < a^{\mathbf{2}}$ and $a \geqslant 0$ holds $-a < b$ and $b < a$.

(7) For all real numbers a, b such that $-a \leqslant b$ and $b \leqslant a$ holds $b^{\mathbf{2}} \leqslant a^{\mathbf{2}}$.

(8) For all real numbers a, b such that $-a < b$ and $b < a$ holds $b^{\mathbf{2}} < a^{\mathbf{2}}$.

In the sequel T, T_1, T_2, S denote non empty topological spaces.

Next we state a number of propositions:

(9) Let f be a map from T_1 into S, g be a map from T_2 into S, and F_1, F_2 be subsets of T. Suppose that T_1 is a subspace of T and T_2 is a subspace of T and $F_1 = \Omega_{(T_1)}$ and $F_2 = \Omega_{(T_2)}$ and $\Omega_{(T_1)} \cup \Omega_{(T_2)} = \Omega_T$ and F_1 is closed and F_2 is closed and f is continuous and g is continuous and for every set p such that $p \in \Omega_{(T_1)} \cap \Omega_{(T_2)}$ holds $f(p) = g(p)$. Then there exists a map h from T into S such that $h = f{+}\!\cdot g$ and h is continuous.

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(10) Let n be a natural number, q_2 be a point of $\mathcal{E}^n$, q be a point of $\mathcal{E}^n_{\mathrm{T}}$, and r be a real number. If $q = q_2$, then $\mathrm{Ball}(q_2, r) = \{q_3; q_3$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $|q - q_3| < r\}$.

(11) $(0_{\mathcal{E}^2_{\mathrm{T}}})_{\mathbf{1}} = 0$ and $(0_{\mathcal{E}^2_{\mathrm{T}}})_{\mathbf{2}} = 0$.

(12) $1.\mathrm{REAL}\,2 = \langle(1$ **qua** real number$), (1$ **qua** real number$)\rangle$.

(13) $(1.\mathrm{REAL}\,2)_{\mathbf{1}} = 1$ and $(1.\mathrm{REAL}\,2)_{\mathbf{2}} = 1$.

(14) $\mathrm{dom}\,\mathrm{proj1} =$ the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and $\mathrm{dom}\,\mathrm{proj1} = \mathcal{R}^2$.

(15) $\mathrm{dom}\,\mathrm{proj2} =$ the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and $\mathrm{dom}\,\mathrm{proj2} = \mathcal{R}^2$.

(16) proj1 is a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$.

(17) proj2 is a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$.

(18) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $p = [\mathrm{proj1}(p), \mathrm{proj2}(p)]$.

(19) For every subset B of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ such that $B = \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ holds $B^{\mathrm{c}} \neq \emptyset$ and (the carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus B \neq \emptyset$.

(20) Let X, Y be non empty topological spaces and f be a map from X into Y. Then f is continuous if and only if for every point p of X and for every subset V of Y such that $f(p) \in V$ and V is open there exists a subset W of X such that $p \in W$ and W is open and $f^{\circ}W \subseteq V$.

(21) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$ and G be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose G is open and $p \in G$. Then there exists a real number r such that $r > 0$ and $\{q; q$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{1}} - r < q_{\mathbf{1}} \wedge q_{\mathbf{1}} < p_{\mathbf{1}} + r \wedge p_{\mathbf{2}} - r < q_{\mathbf{2}} \wedge q_{\mathbf{2}} < p_{\mathbf{2}} + r\} \subseteq G$.

(22) Let X, Y, Z be non empty topological spaces, B be a subset of Y, C be a subset of Z, f be a map from X into Y, and h be a map from $Y \restriction B$ into $Z \restriction C$. Suppose f is continuous and h is continuous and $\mathrm{rng}\, f \subseteq B$ and $B \neq \emptyset$ and $C \neq \emptyset$. Then there exists a map g from X into Z such that g is continuous and $g = h \cdot f$.

In the sequel p, q are points of $\mathcal{E}^2_{\mathrm{T}}$.

The function OutInSq from (the carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ into (the carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ is defined by the condition (Def. 1).

(Def. 1) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$. Then

(i) if $p_{\mathbf{2}} \leqslant p_{\mathbf{1}}$ and $-p_{\mathbf{1}} \leqslant p_{\mathbf{2}}$ or $p_{\mathbf{2}} \geqslant p_{\mathbf{1}}$ and $p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}$, then $\mathrm{OutInSq}(p) = [\frac{1}{p_{\mathbf{1}}}, \frac{\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}}}{p_{\mathbf{1}}}]$, and

(ii) if $p_{\mathbf{2}} \not\leqslant p_{\mathbf{1}}$ or $-p_{\mathbf{1}} \not\leqslant p_{\mathbf{2}}$ and if $p_{\mathbf{2}} \not\geqslant p_{\mathbf{1}}$ or $p_{\mathbf{2}} \not\leqslant -p_{\mathbf{1}}$, then $\mathrm{OutInSq}(p) = [\frac{\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}}}{p_{\mathbf{2}}}, \frac{1}{p_{\mathbf{2}}}]$.

Next we state a number of propositions:

(23) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} \not\leqslant p_{\mathbf{1}}$ or $-p_{\mathbf{1}} \not\leqslant p_{\mathbf{2}}$ but $p_{\mathbf{2}} \not\geqslant p_{\mathbf{1}}$ or $p_{\mathbf{2}} \not\leqslant -p_{\mathbf{1}}$. Then $p_{\mathbf{1}} \leqslant p_{\mathbf{2}}$ and $-p_{\mathbf{2}} \leqslant p_{\mathbf{1}}$ or $p_{\mathbf{1}} \geqslant p_{\mathbf{2}}$ and $p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}$.

(24) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$. Then

(i) if $p_{\mathbf{1}} \leqslant p_{\mathbf{2}}$ and $-p_{\mathbf{2}} \leqslant p_{\mathbf{1}}$ or $p_{\mathbf{1}} \geqslant p_{\mathbf{2}}$ and $p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}$, then $\mathrm{OutInSq}(p) = [\frac{\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}}}{p_{\mathbf{2}}}, \frac{1}{p_{\mathbf{2}}}]$, and

(ii) if $p_{\mathbf{1}} \nleqslant p_{\mathbf{2}}$ or $-p_{\mathbf{2}} \nleqslant p_{\mathbf{1}}$ and if $p_{\mathbf{1}} \ngeqslant p_{\mathbf{2}}$ or $p_{\mathbf{1}} \nleqslant -p_{\mathbf{2}}$, then $\mathrm{OutInSq}(p) = [\frac{1}{p_{\mathbf{1}}}, \frac{\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}}}{p_{\mathbf{1}}}]$.

(25) Let D be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and K_0 be a subset of $(\mathcal{E}^2_{\mathrm{T}})\restriction D$. Suppose $K_0 = \{p : (p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \ \wedge\ -p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \ \vee\ p_{\mathbf{2}} \geqslant p_{\mathbf{1}} \ \wedge\ p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}) \ \wedge\ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then $\mathrm{rng}(\mathrm{OutInSq}\restriction K_0) \subseteq$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction D \restriction K_0$.

(26) Let D be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and K_0 be a subset of $(\mathcal{E}^2_{\mathrm{T}})\restriction D$. Suppose $K_0 = \{p : (p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \ \wedge\ -p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \ \vee\ p_{\mathbf{1}} \geqslant p_{\mathbf{2}} \ \wedge\ p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}) \ \wedge\ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then $\mathrm{rng}(\mathrm{OutInSq}\restriction K_0) \subseteq$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction D \restriction K_0$.

(27) Let K_1 be a set and D be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K_1 = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $(p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \ \wedge\ -p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \ \vee\ p_{\mathbf{2}} \geqslant p_{\mathbf{1}} \ \wedge\ p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}) \ \wedge\ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $D^{\mathrm{c}} = \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then K_1 is a non empty subset of $(\mathcal{E}^2_{\mathrm{T}})\restriction D$ and a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$.

(28) Let K_1 be a set and D be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K_1 = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $(p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \ \wedge\ -p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \ \vee\ p_{\mathbf{1}} \geqslant p_{\mathbf{2}} \ \wedge\ p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}) \ \wedge\ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $D^{\mathrm{c}} = \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then K_1 is a non empty subset of $(\mathcal{E}^2_{\mathrm{T}})\restriction D$ and a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$.

(29) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 + r_2$ and g is continuous.

(30) Let X be a non empty topological space and a be a real number. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X holds $g(p) = a$ and g is continuous.

(31) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 - r_2$ and g is continuous.

(32) Let X be a non empty topological space and f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = r_1 \cdot r_1$ and g is continuous.

(33) Let X be a non empty topological space, f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$, and a be a real number. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = a \cdot r_1$ and g is continuous.

(34) Let X be a non empty topological space, f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$, and a be a real number. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = r_1 + a$ and g is continuous.

(35) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 \cdot r_2$ and g is continuous.

(36) Let X be a non empty topological space and f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and for every point q of X holds $f_1(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = \frac{1}{r_1}$ and g is continuous.

(37) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = \frac{r_1}{r_2}$ and g is continuous.

(38) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that
  (i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = \frac{\frac{r_1}{r_2}}{r_2}$, and
  (ii) g is continuous.

(39) Let K_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_0$ into $\mathbb{R}^{\mathbf{1}}$. If for every point p of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_0$ holds $f(p) = \mathrm{proj1}(p)$, then f is continuous.

(40) Let K_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_0$ into $\mathbb{R}^{\mathbf{1}}$. If for every point p of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_0$ holds $f(p) = \mathrm{proj2}(p)$, then f is continuous.

(41) Let K_2 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_2$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that
  (i) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_2$ holds $f(p) = \frac{1}{p_{\mathbf{1}}}$, and
  (ii) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_2$ holds $q_{\mathbf{1}} \neq 0$.

  Then f is continuous.

(42) Let K_2 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_2$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that
  (i) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_2$ holds $f(p) = \frac{1}{p_{\mathbf{2}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_2$ holds $q_{\mathbf{2}} \neq 0$.

Then f is continuous.

(43) Let K_2 be a non empty subset of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_2$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_2$ holds $f(p) = \frac{\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}}}{p_{\mathbf{1}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_2$ holds $q_{\mathbf{1}} \neq 0$.

Then f is continuous.

(44) Let K_2 be a non empty subset of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_2$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_2$ holds $f(p) = \frac{\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}}}{p_{\mathbf{2}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_2$ holds $q_{\mathbf{2}} \neq 0$.

Then f is continuous.

(45) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$, f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$, and f_1, f_2 be maps from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) f_1 is continuous,

(ii) f_2 is continuous,

(iii) $K_0 \neq \emptyset$,

(iv) $B_0 \neq \emptyset$, and

(v) for all real numbers x, y, r, s such that $[x, y] \in K_0$ and $r = f_1([x, y])$ and $s = f_2([x, y])$ holds $f([x, y]) = [r, s]$.

Then f is continuous.

(46) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{OutInSq}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \wedge -p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \vee p_{\mathbf{2}} \geqslant p_{\mathbf{1}} \wedge p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}) \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(47) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{OutInSq}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \wedge -p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \vee p_{\mathbf{1}} \geqslant p_{\mathbf{2}} \wedge p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}) \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

In this article we present several logical schemes. The scheme *TopSubset* concerns a unary predicate $\mathcal{P}$, and states that:

$\{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $\mathcal{P}[p]\}$ is a subset of $\mathcal{E}^2_{\rm T}$

for all values of the parameters.

The scheme *TopCompl* deals with a subset $\mathcal{A}$ of $\mathcal{E}^2_{\rm T}$ and a unary predicate $\mathcal{P}$, and states that:

$-\mathcal{A} = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: not $\mathcal{P}[p]\}$
provided the parameters meet the following requirement:

- $\mathcal{A} = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $\mathcal{P}[p]\}$.

The scheme *ClosedSubset* deals with two unary functors $\mathcal{F}$ and $\mathcal{G}$ yielding real numbers, and states that:

$\{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $\mathcal{F}(p) \leqslant \mathcal{G}(p)\}$ is a closed subset of $\mathcal{E}^2_{\rm T}$

provided the following conditions are met:

- For all points p, q of $\mathcal{E}^2_{\rm T}$ holds $\mathcal{F}(p-q) = \mathcal{F}(p) - \mathcal{F}(q)$ and $\mathcal{G}(p-q) = \mathcal{G}(p) - \mathcal{G}(q)$, and
- For all points p, q of $\mathcal{E}^2_{\rm T}$ holds $|p-q|^{\mathbf{2}} = |\mathcal{F}(p-q)|^{\mathbf{2}} + |\mathcal{G}(p-q)|^{\mathbf{2}}$.

One can prove the following propositions:

(48) Let B_0 be a subset of $\mathcal{E}^2_{\rm T}$, K_0 be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0 \restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{OutInSq} \restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}) \setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \wedge -p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \vee p_{\mathbf{2}} \geqslant p_{\mathbf{1}} \wedge p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}) \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous and K_0 is closed.

(49) Let B_0 be a subset of $\mathcal{E}^2_{\rm T}$, K_0 be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0 \restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{OutInSq} \restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}) \setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \wedge -p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \vee p_{\mathbf{1}} \geqslant p_{\mathbf{2}} \wedge p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}) \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous and K_0 is closed.

(50) Let D be a non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $D^{\rm c} = \{0_{\mathcal{E}^2_{\rm T}}\}$. Then there exists a map h from $(\mathcal{E}^2_{\rm T})\restriction D$ into $(\mathcal{E}^2_{\rm T})\restriction D$ such that $h = \mathrm{OutInSq}$ and h is continuous.

(51) Let B, K_0, K_3 be subsets of $\mathcal{E}^2_{\rm T}$. Suppose that
(i) $B = \{0_{\mathcal{E}^2_{\rm T}}\}$,
(ii) $K_0 = \{p : -1 < p_{\mathbf{1}} \wedge p_{\mathbf{1}} < 1 \wedge -1 < p_{\mathbf{2}} \wedge p_{\mathbf{2}} < 1\}$, and
(iii) $K_3 = \{q : -1 = q_{\mathbf{1}} \wedge -1 \leqslant q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leqslant 1 \vee q_{\mathbf{1}} = 1 \wedge -1 \leqslant q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leqslant 1 \vee -1 = q_{\mathbf{2}} \wedge -1 \leqslant q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leqslant 1 \vee 1 = q_{\mathbf{2}} \wedge -1 \leqslant q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leqslant 1\}$.
Then there exists a map f from $(\mathcal{E}^2_{\rm T})\restriction B^{\rm c}$ into $(\mathcal{E}^2_{\rm T})\restriction B^{\rm c}$ such that
(iv) f is continuous and one-to-one,
(v) for every point t of $\mathcal{E}^2_{\rm T}$ such that $t \in K_0$ and $t \neq 0_{\mathcal{E}^2_{\rm T}}$ holds $f(t) \notin K_0 \cup K_3$,
(vi) for every point r of $\mathcal{E}^2_{\rm T}$ such that $r \notin K_0 \cup K_3$ holds $f(r) \in K_0$, and
(vii) for every point s of $\mathcal{E}^2_{\rm T}$ such that $s \in K_3$ holds $f(s) = s$.

(52) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, K_0 be a subset of $\mathcal{E}^2_{\rm T}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $K_0 = \{p : -1 < p_{\mathbf{1}} \wedge p_{\mathbf{1}} < 1 \wedge -1 < p_{\mathbf{2}} \wedge p_{\mathbf{2}} < 1\}$ and $f(O)_{\mathbf{1}} = -1$ and $f(I)_{\mathbf{1}} = 1$ and $-1 \leqslant f(O)_{\mathbf{2}}$ and $f(O)_{\mathbf{2}} \leqslant 1$ and $-1 \leqslant f(I)_{\mathbf{2}}$ and $f(I)_{\mathbf{2}} \leqslant 1$ and $g(O)_{\mathbf{2}} = -1$ and $g(I)_{\mathbf{2}} = 1$ and $-1 \leqslant g(O)_{\mathbf{1}}$ and $g(O)_{\mathbf{1}} \leqslant 1$ and $-1 \leqslant g(I)_{\mathbf{1}}$ and $g(I)_{\mathbf{1}} \leqslant 1$ and $\mathrm{rng}\, f \cap K_0 = \emptyset$ and $\mathrm{rng}\, g \cap K_0 = \emptyset$. Then $\mathrm{rng}\, f \cap \mathrm{rng}\, g \neq \emptyset$.

(53) Let A, B, C, D be real numbers and f be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$. Suppose that for every point t of $\mathcal{E}^2_{\rm T}$ holds $f(t) = [A \cdot t_{\mathbf{1}} + B, C \cdot t_{\mathbf{2}} + D]$. Then f is continuous.

(54) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $f(O)_{\mathbf{1}} = a$ and $f(I)_{\mathbf{1}} = b$ and $c \leqslant f(O)_{\mathbf{2}}$ and $f(O)_{\mathbf{2}} \leqslant d$ and $c \leqslant f(I)_{\mathbf{2}}$ and $f(I)_{\mathbf{2}} \leqslant d$ and $g(O)_{\mathbf{2}} = c$ and $g(I)_{\mathbf{2}} = d$ and $a \leqslant g(O)_{\mathbf{1}}$ and $g(O)_{\mathbf{1}} \leqslant b$ and $a \leqslant g(I)_{\mathbf{1}}$ and $g(I)_{\mathbf{1}} \leqslant b$ and $a < b$ and $c < d$ and it is not true that there exists a point r of $\mathbb{I}$ such that $a < f(r)_{\mathbf{1}}$ and $f(r)_{\mathbf{1}} < b$ and $c < f(r)_{\mathbf{2}}$ and $f(r)_{\mathbf{2}} < d$ and it is not true that there exists a point r of $\mathbb{I}$ such that $a < g(r)_{\mathbf{1}}$ and $g(r)_{\mathbf{1}} < b$ and $c < g(r)_{\mathbf{2}}$ and $g(r)_{\mathbf{2}} < d$. Then $\operatorname{rng} f \cap \operatorname{rng} g \neq \emptyset$.

(55)(i) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\rm T}$: $(p_7)_{\mathbf{2}} \leqslant (p_7)_{\mathbf{1}}\}$ is a closed subset of $\mathcal{E}^2_{\rm T}$, and

(ii) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\rm T}$: $(p_7)_{\mathbf{1}} \leqslant (p_7)_{\mathbf{2}}\}$ is a closed subset of $\mathcal{E}^2_{\rm T}$.

(56)(i) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\rm T}$: $-(p_7)_{\mathbf{1}} \leqslant (p_7)_{\mathbf{2}}\}$ is a closed subset of $\mathcal{E}^2_{\rm T}$, and

(ii) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\rm T}$: $(p_7)_{\mathbf{2}} \leqslant -(p_7)_{\mathbf{1}}\}$ is a closed subset of $\mathcal{E}^2_{\rm T}$.

(57)(i) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\rm T}$: $-(p_7)_{\mathbf{2}} \leqslant (p_7)_{\mathbf{1}}\}$ is a closed subset of $\mathcal{E}^2_{\rm T}$, and

(ii) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\rm T}$: $(p_7)_{\mathbf{1}} \leqslant -(p_7)_{\mathbf{2}}\}$ is a closed subset of $\mathcal{E}^2_{\rm T}$.

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