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On the Simple Closed Curve Property of the Circle and the Fashoda Meet Theorem

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Summary. First, we prove the fact that the circle is the simple closed curve, which was defined as a curve homeomorphic to the square. For this proof, we introduce a mapping which is a homeomorphism from 2-dimensional plane to itself. This mapping maps the square to the circle. Secondly, we prove the Fashoda meet theorem for the circle using this homeomorphism.

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The terminology and notation used in this paper have been introduced in the following articles: [17], [5], [7], [1], [2], [11], [3], [12], [4], [13], [10], [18], [15], [16], [14], [8], [9], and [6].

1. Preliminaries

In this paper x, y, z, u, a are real numbers.

We now state a number of propositions:

(1) If $x^2 = y^2$, then $x = y$ or $x = -y$.

(2) If $x^2 = 1$, then $x = 1$ or $x = -1$.

(3) If $0 \leqslant x$ and $x \leqslant 1$, then $x^2 \leqslant x$.

(4) If $a \geqslant 0$ and $(x - a) \cdot (x + a) \leqslant 0$, then $-a \leqslant x$ and $x \leqslant a$.

(5) If $x^2 - 1 \leqslant 0$, then $-1 \leqslant x$ and $x \leqslant 1$.

(6) $x < y$ and $x < z$ iff $x < \min(y, z)$.

(7) If $0 < x$, then $\frac{x}{3} < x$ and $\frac{x}{4} < x$.

(8) If $x \geqslant 1$, then $\sqrt{x} \geqslant 1$ and if $x > 1$, then $\sqrt{x} > 1$.

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(9) If $x \leqslant y$ and $z \leqslant u$, then $[y, z] \subseteq [x, u]$.

(10) For every point p of $\mathcal{E}^2_{\rm T}$ holds $|p| = \sqrt{(p_{\mathbf{1}})^{\mathbf{2}} + (p_{\mathbf{2}})^{\mathbf{2}}}$ and $|p|^{\mathbf{2}} = (p_{\mathbf{1}})^{\mathbf{2}} + (p_{\mathbf{2}})^{\mathbf{2}}$.

(11) For every function f and for all sets B, C holds $(f{\restriction}B)^{\circ}C = f^{\circ}(C \cap B)$.

(12) Let X be a topological structure, Y be a non empty topological structure, f be a map from X into Y, and P be a subset of X. Then $f{\restriction}P$ is a map from $X{\restriction}P$ into Y.

(13) Let X, Y be non empty topological spaces, p_0 be a point of X, D be a non empty subset of X, E be a non empty subset of Y, and f be a map from X into Y. Suppose that $D^{\rm c} = \{p_0\}$ and $E^{\rm c} = \{f(p_0)\}$ and X is a T_2 space and Y is a T_2 space and for every point p of $X{\restriction}D$ holds $f(p) \neq f(p_0)$ and there exists a map h from $X{\restriction}D$ into $Y{\restriction}E$ such that $h = f{\restriction}D$ and h is continuous and for every subset V of Y such that $f(p_0) \in V$ and V is open there exists a subset W of X such that $p_0 \in W$ and W is open and $f^{\circ}W \subseteq V$. Then f is continuous.

2. The Circle is a Simple Closed Curve

In the sequel p, q denote points of $\mathcal{E}^2_{\rm T}$.

The function SqCirc from the carrier of $\mathcal{E}^2_{\rm T}$ into the carrier of $\mathcal{E}^2_{\rm T}$ is defined by the condition (Def. 1).

(Def. 1) Let p be a point of $\mathcal{E}^2_{\rm T}$. Then

(i) if $p = 0_{\mathcal{E}^2_{\rm T}}$, then $\mathrm{SqCirc}(p) = p$,

(ii) if $p_{\mathbf{2}} \leqslant p_{\mathbf{1}}$ and $-p_{\mathbf{1}} \leqslant p_{\mathbf{2}}$ or $p_{\mathbf{2}} \geqslant p_{\mathbf{1}}$ and $p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}$ and if $p \neq 0_{\mathcal{E}^2_{\rm T}}$, then $\mathrm{SqCirc}(p) = [\frac{p_{\mathbf{1}}}{\sqrt{1+(\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}}, \frac{p_{\mathbf{2}}}{\sqrt{1+(\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}}]$, and

(iii) if $p_{\mathbf{2}} \not\leqslant p_{\mathbf{1}}$ or $-p_{\mathbf{1}} \not\leqslant p_{\mathbf{2}}$ but $p_{\mathbf{2}} \not\geqslant p_{\mathbf{1}}$ or $p_{\mathbf{2}} \not\leqslant -p_{\mathbf{1}}$ and $p \neq 0_{\mathcal{E}^2_{\rm T}}$, then $\mathrm{SqCirc}(p) = [\frac{p_{\mathbf{1}}}{\sqrt{1+(\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}}, \frac{p_{\mathbf{2}}}{\sqrt{1+(\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}}]$.

We now state a number of propositions:

(14) Let p be a point of $\mathcal{E}^2_{\rm T}$ such that $p \neq 0_{\mathcal{E}^2_{\rm T}}$. Then

(i) if $p_{\mathbf{1}} \leqslant p_{\mathbf{2}}$ and $-p_{\mathbf{2}} \leqslant p_{\mathbf{1}}$ or $p_{\mathbf{1}} \geqslant p_{\mathbf{2}}$ and $p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}$, then $\mathrm{SqCirc}(p) = [\frac{p_{\mathbf{1}}}{\sqrt{1+(\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}}, \frac{p_{\mathbf{2}}}{\sqrt{1+(\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}}]$, and

(ii) if $p_{\mathbf{1}} \not\leqslant p_{\mathbf{2}}$ or $-p_{\mathbf{2}} \not\leqslant p_{\mathbf{1}}$ and if $p_{\mathbf{1}} \not\geqslant p_{\mathbf{2}}$ or $p_{\mathbf{1}} \not\leqslant -p_{\mathbf{2}}$, then $\mathrm{SqCirc}(p) = [\frac{p_{\mathbf{1}}}{\sqrt{1+(\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}}, \frac{p_{\mathbf{2}}}{\sqrt{1+(\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}}]$.

(15) Let X be a non empty topological space and f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and for every point q of X there exists a real number r such that $f_1(q) = r$ and $r \geqslant 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = \sqrt{r_1}$ and g is continuous.

(16) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = (\frac{r_1}{r_2})^{\mathbf{2}}$, and

(ii) g is continuous.

(17) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = 1 + (\frac{r_1}{r_2})^{\mathbf{2}}$, and

(ii) g is continuous.

(18) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = \sqrt{1 + (\frac{r_1}{r_2})^{\mathbf{2}}}$, and

(ii) g is continuous.

(19) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = \frac{r_1}{\sqrt{1+(\frac{r_1}{r_2})^{\mathbf{2}}}}$, and

(ii) g is continuous.

(20) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = \frac{r_2}{\sqrt{1+(\frac{r_1}{r_2})^{\mathbf{2}}}}$, and

(ii) g is continuous.

(21) Let K_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $f(p) = \frac{p_{\mathbf{1}}}{\sqrt{1+(\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $q_{\mathbf{1}} \neq 0$.

Then f is continuous.

(22) Let K_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $f(p) = \frac{p_{\bf 2}}{\sqrt{1+(\frac{p_{\bf 2}}{p_{\bf 1}})^{\bf 2}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $q_{\bf 1} \neq 0$.

Then f is continuous.

(23) Let K_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_1$ into $\mathbb{R}^{\bf 1}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $f(p) = \frac{p_{\bf 2}}{\sqrt{1+(\frac{p_{\bf 1}}{p_{\bf 2}})^{\bf 2}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $q_{\bf 2} \neq 0$.

Then f is continuous.

(24) Let K_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_1$ into $\mathbb{R}^{\bf 1}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $f(p) = \frac{p_{\bf 1}}{\sqrt{1+(\frac{p_{\bf 1}}{p_{\bf 2}})^{\bf 2}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $q_{\bf 2} \neq 0$.

Then f is continuous.

(25) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $f = \text{SqCirc} \restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\bf 2} \leqslant p_{\bf 1} \ \wedge \ -p_{\bf 1} \leqslant p_{\bf 2} \ \vee \ p_{\bf 2} \geqslant p_{\bf 1} \ \wedge \ p_{\bf 2} \leqslant -p_{\bf 1}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(26) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $f = \text{SqCirc} \restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\bf 1} \leqslant p_{\bf 2} \ \wedge \ -p_{\bf 2} \leqslant p_{\bf 1} \ \vee \ p_{\bf 1} \geqslant p_{\bf 2} \ \wedge \ p_{\bf 1} \leqslant -p_{\bf 2}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

In this article we present several logical schemes. The scheme *TopIncl* concerns a unary predicate $\mathcal{P}$, and states that:

$$\{p : \mathcal{P}[p] \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\} \subseteq (\text{the carrier of } \mathcal{E}^2_{\rm T}) \setminus \{0_{\mathcal{E}^2_{\rm T}}\}$$

for all values of the parameters.

The scheme *TopInter* concerns a unary predicate $\mathcal{P}$, and states that:

$$\{p : \mathcal{P}[p] \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\} = \{p_7; p_7 \text{ ranges over points of } \mathcal{E}^2_{\rm T}\text{: } \mathcal{P}[p_7]\} \cap ((\text{the carrier of } \mathcal{E}^2_{\rm T}) \setminus \{0_{\mathcal{E}^2_{\rm T}}\})$$

for all values of the parameters.

Next we state several propositions:

(27) Let B_0 be a subset of $\mathcal{E}^2_{\rm T}$, K_0 be a subset of $(\mathcal{E}^2_{\rm T}) \restriction B_0$, and f be a map from $(\mathcal{E}^2_{\rm T}) \restriction B_0 \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $f = \text{SqCirc} \restriction K_0$ and $B_0 =$ (the

carrier of $\mathcal{E}^2_{\mathrm{T}}$) $\setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : (p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \wedge -p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \vee p_{\mathbf{2}} \geqslant p_{\mathbf{1}} \wedge p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}) \wedge p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then f is continuous and K_0 is closed.

(28) Let B_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$, K_0 be a subset of $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$. Suppose $f = \mathrm{SqCirc}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}$) $\setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : (p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \wedge -p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \vee p_{\mathbf{1}} \geqslant p_{\mathbf{2}} \wedge p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}) \wedge p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then f is continuous and K_0 is closed.

(29) Let D be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $D^{\mathrm{c}} = \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then there exists a map h from $(\mathcal{E}^2_{\mathrm{T}})\restriction D$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction D$ such that $h = \mathrm{SqCirc}\restriction D$ and h is continuous.

(30) For every non empty subset D of $\mathcal{E}^2_{\mathrm{T}}$ such that $D =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}$) $\setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ holds $D^{\mathrm{c}} = \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$.

(31) There exists a map h from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that $h = \mathrm{SqCirc}$ and h is continuous.

(32) SqCirc is one-to-one.

Let us observe that SqCirc is one-to-one.

One can prove the following propositions:

(33) Let K_2, C_1 be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that
 (i) $K_2 = \{q : -1 = q_{\mathbf{1}} \wedge -1 \leqslant q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leqslant 1 \vee q_{\mathbf{1}} = 1 \wedge -1 \leqslant q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leqslant 1 \vee -1 = q_{\mathbf{2}} \wedge -1 \leqslant q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leqslant 1 \vee 1 = q_{\mathbf{2}} \wedge -1 \leqslant q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leqslant 1\}$, and
 (ii) $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p_2| = 1\}$.

 Then $\mathrm{SqCirc}^\circ K_2 = C_1$.

(34) Let P, K_2 be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_2$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction P$. Suppose that
 (i) $K_2 = \{q : -1 = q_{\mathbf{1}} \wedge -1 \leqslant q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leqslant 1 \vee q_{\mathbf{1}} = 1 \wedge -1 \leqslant q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leqslant 1 \vee -1 = q_{\mathbf{2}} \wedge -1 \leqslant q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leqslant 1 \vee 1 = q_{\mathbf{2}} \wedge -1 \leqslant q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leqslant 1\}$, and
 (ii) f is a homeomorphism.

 Then P is a simple closed curve.

(35) Let K_2 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K_2 = \{q : -1 = q_{\mathbf{1}} \wedge -1 \leqslant q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leqslant 1 \vee q_{\mathbf{1}} = 1 \wedge -1 \leqslant q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leqslant 1 \vee -1 = q_{\mathbf{2}} \wedge -1 \leqslant q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leqslant 1 \vee 1 = q_{\mathbf{2}} \wedge -1 \leqslant q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leqslant 1\}$. Then K_2 is a simple closed curve and compact.

(36) For every subset C_1 of $\mathcal{E}^2_{\mathrm{T}}$ such that $C_1 = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ holds C_1 is a simple closed curve.

3. The Fashoda Meet Theorem for the Circle

Next we state a number of propositions:

(37) Let K_0, C_0 be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K_0 = \{p : -1 \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant 1 \wedge -1 \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant 1\}$ and $C_0 = \{p_1; p_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p_1| \leqslant 1\}$. Then $\mathrm{SqCirc}^{-1}(C_0) \subseteq K_0$.

(38) Let given p. Then

(i) if $p = 0_{\mathcal{E}^2_{\mathrm{T}}}$, then $\mathrm{SqCirc}^{-1}(p) = 0_{\mathcal{E}^2_{\mathrm{T}}}$,

(ii) if $p_{\mathbf{2}} \leqslant p_{\mathbf{1}}$ and $-p_{\mathbf{1}} \leqslant p_{\mathbf{2}}$ or $p_{\mathbf{2}} \geqslant p_{\mathbf{1}}$ and $p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}$ and if $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$, then $\mathrm{SqCirc}^{-1}(p) = [p_{\mathbf{1}} \cdot \sqrt{1 + (\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}, p_{\mathbf{2}} \cdot \sqrt{1 + (\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}]$, and

(iii) if $p_{\mathbf{2}} \not\leqslant p_{\mathbf{1}}$ or $-p_{\mathbf{1}} \not\leqslant p_{\mathbf{2}}$ but $p_{\mathbf{2}} \not\geqslant p_{\mathbf{1}}$ or $p_{\mathbf{2}} \not\leqslant -p_{\mathbf{1}}$ and $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$, then $\mathrm{SqCirc}^{-1}(p) = [p_{\mathbf{1}} \cdot \sqrt{1 + (\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}, p_{\mathbf{2}} \cdot \sqrt{1 + (\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}]$.

(39) SqCirc^{-1} is a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$.

(40) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$. Then

(i) if $p_{\mathbf{1}} \leqslant p_{\mathbf{2}}$ and $-p_{\mathbf{2}} \leqslant p_{\mathbf{1}}$ or $p_{\mathbf{1}} \geqslant p_{\mathbf{2}}$ and $p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}$, then $\mathrm{SqCirc}^{-1}(p) = [p_{\mathbf{1}} \cdot \sqrt{1 + (\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}, p_{\mathbf{2}} \cdot \sqrt{1 + (\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}]$, and

(ii) if $p_{\mathbf{1}} \not\leqslant p_{\mathbf{2}}$ or $-p_{\mathbf{2}} \not\leqslant p_{\mathbf{1}}$ and if $p_{\mathbf{1}} \not\geqslant p_{\mathbf{2}}$ or $p_{\mathbf{1}} \not\leqslant -p_{\mathbf{2}}$, then $\mathrm{SqCirc}^{-1}(p) = [p_{\mathbf{1}} \cdot \sqrt{1 + (\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}, p_{\mathbf{2}} \cdot \sqrt{1 + (\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}]$.

(41) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 \cdot \sqrt{1 + (\frac{r_1}{r_2})^{\mathbf{2}}}$, and

(ii) g is continuous.

(42) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_2 \cdot \sqrt{1 + (\frac{r_1}{r_2})^{\mathbf{2}}}$, and

(ii) g is continuous.

(43) Let K_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $f(p) = p_{\mathbf{1}} \cdot \sqrt{1 + (\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $q_{\mathbf{1}} \neq 0$.

Then f is continuous.

(44) Let K_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $f(p) = p_{\mathbf{2}} \cdot \sqrt{1 + (\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}})^{\mathbf{2}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $q_{\mathbf{1}} \neq 0$.

Then f is continuous.

(45) Let K_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $f(p) = p_{\mathbf{2}} \cdot \sqrt{1 + (\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $q_{\mathbf{2}} \neq 0$.

Then f is continuous.

(46) Let K_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $f(p) = p_{\mathbf{1}} \cdot \sqrt{1 + (\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}})^{\mathbf{2}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $q_{\mathbf{2}} \neq 0$.

Then f is continuous.

(47) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{SqCirc}^{-1}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \ \wedge \ -p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \ \vee \ p_{\mathbf{2}} \geqslant p_{\mathbf{1}} \ \wedge \ p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(48) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$ and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{SqCirc}^{-1}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \ \wedge \ -p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \ \vee \ p_{\mathbf{1}} \geqslant p_{\mathbf{2}} \ \wedge \ p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(49) Let B_0 be a subset of $\mathcal{E}^2_{\rm T}$, K_0 be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{SqCirc}^{-1}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \ \wedge \ -p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \ \vee \ p_{\mathbf{2}} \geqslant p_{\mathbf{1}} \ \wedge \ p_{\mathbf{2}} \leqslant -p_{\mathbf{1}}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous and K_0 is closed.

(50) Let B_0 be a subset of $\mathcal{E}^2_{\rm T}$, K_0 be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $f = \mathrm{SqCirc}^{-1}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : (p_{\mathbf{1}} \leqslant p_{\mathbf{2}} \ \wedge \ -p_{\mathbf{2}} \leqslant p_{\mathbf{1}} \ \vee \ p_{\mathbf{1}} \geqslant p_{\mathbf{2}} \ \wedge \ p_{\mathbf{1}} \leqslant -p_{\mathbf{2}}) \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous and K_0 is closed.

(51) Let D be a non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $D^{\rm c} = \{0_{\mathcal{E}^2_{\rm T}}\}$. Then there exists a map h from $(\mathcal{E}^2_{\rm T})\restriction D$ into $(\mathcal{E}^2_{\rm T})\restriction D$ such that $h = \mathrm{SqCirc}^{-1}\restriction D$ and h is continuous.

(52) There exists a map h from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ such that $h = \mathrm{SqCirc}^{-1}$ and h is continuous.

(54)[1](i) SqCirc is a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$,

(ii) rng SqCirc = the carrier of $\mathcal{E}^2_{\rm T}$, and

[1]The proposition (53) has been removed.

(iii) for every map f from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ such that $f = \mathrm{SqCirc}$ holds f is a homeomorphism.

(55) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, C_0, K_3, K_4, K_5, K_6 be subsets of $\mathcal{E}^2_{\rm T}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leqslant 1\}$ and $K_3 = \{q_1; q_1$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_1| = 1 \wedge (q_1)_{\mathbf{2}} \leqslant (q_1)_{\mathbf{1}} \wedge (q_1)_{\mathbf{2}} \geqslant -(q_1)_{\mathbf{1}}\}$ and $K_4 = \{q_2; q_2$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_2| = 1 \wedge (q_2)_{\mathbf{2}} \geqslant (q_2)_{\mathbf{1}} \wedge (q_2)_{\mathbf{2}} \leqslant -(q_2)_{\mathbf{1}}\}$ and $K_5 = \{q_3; q_3$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_3| = 1 \wedge (q_3)_{\mathbf{2}} \geqslant (q_3)_{\mathbf{1}} \wedge (q_3)_{\mathbf{2}} \geqslant -(q_3)_{\mathbf{1}}\}$ and $K_6 = \{q_4; q_4$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_4| = 1 \wedge (q_4)_{\mathbf{2}} \leqslant (q_4)_{\mathbf{1}} \wedge (q_4)_{\mathbf{2}} \leqslant -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_4$ and $f(I) \in K_3$ and $g(O) \in K_6$ and $g(I) \in K_5$ and $\mathrm{rng}\, f \subseteq C_0$ and $\mathrm{rng}\, g \subseteq C_0$. Then $\mathrm{rng}\, f \cap \mathrm{rng}\, g \neq \emptyset$.

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