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More on the External Approximation of a Continuum[1]

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Summary. The main goal was to prove two facts:

- the gauge is the Go-board of a corresponding cage,
- the left components of the complement of the curve determined by a cage are monotonic w.r.t. the index of the approximation.

Some auxiliary facts are proved, too. At the end new notions needed for internal approximation are defined and some useful lemmas are proved.

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The terminology and notation used in this paper have been introduced in the following articles: [28], [40], [1], [3], [12], [29], [14], [4], [5], [37], [33], [13], [6], [20], [21], [26], [32], [9], [35], [24], [18], [27], [25], [8], [11], [17], [2], [36], [38], [30], [10], [16], [41], [43], [42], [19], [23], [34], [39], [31], [15], [44], [22], and [7].

1. Preliminaries

For simplicity, we follow the rules: m, k, j, j_1, i, i_1, i_2, n are natural numbers, r, s, r_1, t are real numbers, C, D are compact non vertical non horizontal non empty subsets of $\mathcal{E}^2_{\rm T}$, f is a finite sequence of elements of the carrier of $\mathcal{E}^2_{\rm T}$, G is a Go-board, and p is a point of $\mathcal{E}^2_{\rm T}$.

We now state three propositions:

(1) For all sets A, x, y such that A meets $\{x, y\}$ holds $x \in A$ or $y \in A$.

(2) If $r < 0$ and $r_1 \leqslant r$ and $0 \leqslant t$, then $\frac{t}{r} \leqslant \frac{t}{r_1}$.

(3) For every set X and for every binary relation R such that R is reflexive in X holds $X \subseteq \operatorname{field} R$.

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Let us observe that there exists a set which has a non-empty element.

Let D be a non empty set with a non-empty element. Observe that there exists a finite sequence of elements of D^* which is non empty and non-empty.

Let D be a non empty set with non empty elements. One can check that there exists a finite sequence of elements of D^* which is non empty and non-empty.

Let F be a non-empty function yielding function. Note that $\operatorname{rng}_\kappa F(\kappa)$ is non-empty.

Let us note that every finite sequence of elements of $\mathbb{R}$ which is increasing is also one-to-one.

One can prove the following propositions:

(4) For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathcal{L}(p,q) \setminus \{p,q\}$ is convex.

(5) For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathcal{L}(p,q) \setminus \{p,q\}$ is connected.

(6) For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq q$ holds $p \in \overline{\mathcal{L}(p,q) \setminus \{p,q\}}$.

(7) For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq q$ holds $\overline{\mathcal{L}(p,q) \setminus \{p,q\}} = \mathcal{L}(p,q)$.

(8) Let S be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p \neq q$ and $\mathcal{L}(p,q) \setminus \{p,q\} \subseteq S$, then $\mathcal{L}(p,q) \subseteq \overline{S}$.

2. Transforming Finite Sets to Finite Sequences

The binary relation RealOrd on $\mathbb{R}$ is defined by:

(Def. 1) $\mathrm{RealOrd} = \{\langle r,\, s\rangle : r \leqslant s\}$.

Next we state two propositions:

(9) If $\langle r,\, s\rangle \in \mathrm{RealOrd}$, then $r \leqslant s$.

(10) $\operatorname{field} \mathrm{RealOrd} = \mathbb{R}$.

Let us note that RealOrd is ordering and linear-order.

The following propositions are true:

(11) RealOrd linearly orders $\mathbb{R}$.

(12) For every finite subset A of $\mathbb{R}$ holds $\mathrm{SgmX}(\mathrm{RealOrd}, A)$ is increasing.

(13) For every finite sequence f of elements of $\mathbb{R}$ and for every finite subset A of $\mathbb{R}$ such that $A = \operatorname{rng} f$ holds $\mathrm{SgmX}(\mathrm{RealOrd}, A) = \mathrm{Inc}(f)$.

Let A be a finite subset of $\mathbb{R}$. One can verify that $\mathrm{SgmX}(\mathrm{RealOrd}, A)$ is increasing.

Next we state two propositions:

(14) Let X be a non empty set, A be a finite subset of X, and R be an order in X. If R linearly orders A, then $\operatorname{len} \mathrm{SgmX}(R, A) = \operatorname{card} A$.

(15) For every non empty set X and for every finite subset A of X and for every linear-order order R in X holds $\operatorname{len} \mathrm{SgmX}(R, A) = \operatorname{card} A$.

3. On the Construction of Go-boards

Next we state two propositions:

(16) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathbf{X}$-coordinate$(f) = \mathrm{proj1} \cdot f$.

(17) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathbf{Y}$-coordinate$(f) = \mathrm{proj2} \cdot f$.

Let D be a non empty set and let M be a finite sequence of elements of D^*. Then $\mathrm{Values}\, M$ is a subset of D.

Let D be a non empty set with non empty elements and let M be a non empty non-empty finite sequence of elements of D^*. One can verify that $\mathrm{Values}\, M$ is non empty.

The following propositions are true:

(18) For every non empty set D and for every matrix M over D and for every i such that $i \in \mathrm{Seg}\,\mathrm{width}\, M$ holds $\mathrm{rng}(M_{\square,i}) \subseteq \mathrm{Values}\, M$.

(19) For every non empty set D and for every matrix M over D and for every i such that $i \in \mathrm{dom}\, M$ holds $\mathrm{rng}\,\mathrm{Line}(M, i) \subseteq \mathrm{Values}\, M$.

(20) For every column $\mathbf{X}$-increasing non empty yielding matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{len}\, G \leqslant \mathrm{card}(\mathrm{proj1}^\circ\, \mathrm{Values}\, G)$.

(21) For every line $\mathbf{X}$-constant matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{card}(\mathrm{proj1}^\circ\, \mathrm{Values}\, G) \leqslant \mathrm{len}\, G$.

(22) For every line $\mathbf{X}$-constant column $\mathbf{X}$-increasing non empty yielding matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{len}\, G = \mathrm{card}(\mathrm{proj1}^\circ\, \mathrm{Values}\, G)$.

(23) For every line $\mathbf{Y}$-increasing non empty yielding matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{width}\, G \leqslant \mathrm{card}(\mathrm{proj2}^\circ\, \mathrm{Values}\, G)$.

(24) For every column $\mathbf{Y}$-constant non empty yielding matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{card}(\mathrm{proj2}^\circ\, \mathrm{Values}\, G) \leqslant \mathrm{width}\, G$.

(25) For every column $\mathbf{Y}$-constant line $\mathbf{Y}$-increasing non empty yielding matrix G over $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{width}\, G = \mathrm{card}(\mathrm{proj2}^\circ\, \mathrm{Values}\, G)$.

4. More about Go-boards

Next we state several propositions:

(26) For every standard special circular sequence f such that $1 \leqslant k$ and $k + 1 \leqslant \mathrm{len}\, f$ holds $\mathcal{L}(f, k) \subseteq \mathrm{leftcell}(f, k)$.

(27) For every standard special circular sequence f such that $1 \leqslant k$ and $k + 1 \leqslant \mathrm{len}\, f$ holds $\mathrm{left\_cell}(f, k, \text{the Go-board of } f) = \mathrm{leftcell}(f, k)$.

(28) For every standard special circular sequence f such that $1 \leqslant k$ and $k + 1 \leqslant \mathrm{len}\, f$ holds $\mathcal{L}(f, k) \subseteq \mathrm{rightcell}(f, k)$.

(29) For every standard special circular sequence f such that $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$ holds right_cell(f, k, the Go-board of f) = rightcell(f, k).

(30) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a non constant standard special circular sequence. If P is a component of $(\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$, then $P = \mathrm{RightComp}(f)$ or $P = \mathrm{LeftComp}(f)$.

(31) Let f be a non constant standard special circular sequence. Suppose f is a sequence which elements belong to G. Let given k. If $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$, then Int right_cell$(f, k, G) \subseteq \mathrm{RightComp}(f)$ and Int left_cell$(f, k, G) \subseteq \mathrm{LeftComp}(f)$.

(32) Let i_1, j_1, i_2, j_2 be natural numbers and G be a Go-board. Suppose $\langle i_1, j_1\rangle \in$ the indices of G and $\langle i_2, j_2\rangle \in$ the indices of G and $G \circ (i_1, j_1) = G \circ (i_2, j_2)$. Then $i_1 = i_2$ and $j_1 = j_2$.

(33) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and M be a Go-board. Suppose f is a sequence which elements belong to M. Then $\mathrm{mid}(f, i_1, i_2)$ is a sequence which elements belong to M.

Let us mention that every Go-board is non empty and non-empty.

The following propositions are true:

(34) For every Go-board G such that $1 \leqslant i$ and $i \leqslant \operatorname{len} G$ holds $(\mathrm{SgmX}(\mathrm{RealOrd}, \mathrm{proj1}^{\circ} \operatorname{Values} G))(i) = (G \circ (i, 1))_{\mathbf{1}}$.

(35) For every Go-board G such that $1 \leqslant j$ and $j \leqslant \operatorname{width} G$ holds $(\mathrm{SgmX}(\mathrm{RealOrd}, \mathrm{proj2}^{\circ} \operatorname{Values} G))(j) = (G \circ (1, j))_{\mathbf{2}}$.

(36) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and G be a Go-board. Suppose that

(i) f is a sequence which elements belong to G,

(ii) there exists i such that $\langle 1, i\rangle \in$ the indices of G and $G \circ (1, i) \in \operatorname{rng} f$, and

(iii) there exists i such that $\langle \operatorname{len} G, i\rangle \in$ the indices of G and $G \circ (\operatorname{len} G, i) \in \operatorname{rng} f$.

Then $\mathrm{proj1}^{\circ} \operatorname{rng} f = \mathrm{proj1}^{\circ} \operatorname{Values} G$.

(37) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and G be a Go-board. Suppose that

(i) f is a sequence which elements belong to G,

(ii) there exists i such that $\langle i, 1\rangle \in$ the indices of G and $G \circ (i, 1) \in \operatorname{rng} f$, and

(iii) there exists i such that $\langle i, \operatorname{width} G\rangle \in$ the indices of G and $G \circ (i, \operatorname{width} G) \in \operatorname{rng} f$.

Then $\mathrm{proj2}^{\circ} \operatorname{rng} f = \mathrm{proj2}^{\circ} \operatorname{Values} G$.

Let G be a Go-board. Observe that $\operatorname{Values} G$ is non empty.

One can prove the following three propositions:

(38) For every Go-board G holds $G =$ the Go-board of SgmX(RealOrd,

$\mathrm{proj1}^{\circ}\,\mathrm{Values}\,G)$, $\mathrm{SgmX}(\mathrm{RealOrd}, \mathrm{proj2}^{\circ}\,\mathrm{Values}\,G)$.

(39) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and G be a Go-board. If $\mathrm{proj1}^{\circ}\,\mathrm{rng}\,f = \mathrm{proj1}^{\circ}\,\mathrm{Values}\,G$ and $\mathrm{proj2}^{\circ}\,\mathrm{rng}\,f = \mathrm{proj2}^{\circ}\,\mathrm{Values}\,G$, then $G =$ the Go-board of f.

(40) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and G be a Go-board. Suppose that

(i) f is a sequence which elements belong to G,

(ii) there exists i such that $\langle 1, i\rangle \in$ the indices of G and $G \circ (1, i) \in \mathrm{rng}\,f$,

(iii) there exists i such that $\langle i, 1\rangle \in$ the indices of G and $G \circ (i, 1) \in \mathrm{rng}\,f$,

(iv) there exists i such that $\langle \mathrm{len}\,G, i\rangle \in$ the indices of G and $G \circ (\mathrm{len}\,G, i) \in \mathrm{rng}\,f$, and

(v) there exists i such that $\langle i, \mathrm{width}\,G\rangle \in$ the indices of G and $G \circ (i, \mathrm{width}\,G) \in \mathrm{rng}\,f$.

Then $G =$ the Go-board of f.

5. More about Gauges

The following propositions are true:

(41) If $m \leqslant n$ and $1 \leqslant i$ and $i+1 \leqslant \mathrm{len}\,\mathrm{Gauge}(C, n)$, then $\lfloor \frac{i-2}{2^{n-'m}} + 2 \rfloor$ is a natural number.

(42) If $m \leqslant n$ and $1 \leqslant i$ and $i+1 \leqslant \mathrm{len}\,\mathrm{Gauge}(C, n)$, then $1 \leqslant \lfloor \frac{i-2}{2^{n-'m}} + 2 \rfloor$ and $\lfloor \frac{i-2}{2^{n-'m}} + 2 \rfloor + 1 \leqslant \mathrm{len}\,\mathrm{Gauge}(C, m)$.

(43) Suppose $m \leqslant n$ and $1 \leqslant i$ and $i+1 \leqslant \mathrm{len}\,\mathrm{Gauge}(C, n)$ and $1 \leqslant j$ and $j+1 \leqslant \mathrm{width}\,\mathrm{Gauge}(C, n)$. Then there exist i_1, j_1 such that $i_1 = \lfloor \frac{i-2}{2^{n-'m}} + 2 \rfloor$ and $j_1 = \lfloor \frac{j-2}{2^{n-'m}} + 2 \rfloor$ and $\mathrm{cell}(\mathrm{Gauge}(C, n), i, j) \subseteq \mathrm{cell}(\mathrm{Gauge}(C, m), i_1, j_1)$.

(44) Suppose $m \leqslant n$ and $1 \leqslant i$ and $i+1 \leqslant \mathrm{len}\,\mathrm{Gauge}(C, n)$ and $1 \leqslant j$ and $j+1 \leqslant \mathrm{width}\,\mathrm{Gauge}(C, n)$. Then there exist i_1, j_1 such that $1 \leqslant i_1$ and $i_1+1 \leqslant \mathrm{len}\,\mathrm{Gauge}(C, m)$ and $1 \leqslant j_1$ and $j_1+1 \leqslant \mathrm{width}\,\mathrm{Gauge}(C, m)$ and $\mathrm{cell}(\mathrm{Gauge}(C, n), i, j) \subseteq \mathrm{cell}(\mathrm{Gauge}(C, m), i_1, j_1)$.

(45) If $i \leqslant \mathrm{len}\,\mathrm{Gauge}(C, n)$, then $\mathrm{cell}(\mathrm{Gauge}(C, n), i, 0) \subseteq \mathrm{UBD}\,C$.

(46) If $i \leqslant \mathrm{len}\,\mathrm{Gauge}(C, n)$, then $\mathrm{cell}(\mathrm{Gauge}(C, n), i, \mathrm{width}\,\mathrm{Gauge}(C, n)) \subseteq \mathrm{UBD}\,C$.

(47) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that P is Bounded holds $\mathrm{UBD}\,P$ is not Bounded.

(48) Let f be a non constant standard special circular sequence. If $f^{p}_{\circlearrowleft}$ is clockwise oriented, then f is clockwise oriented.

(49) For every non constant standard special circular sequence f such that $\mathrm{LeftComp}(f) = \mathrm{UBD}\,\widetilde{\mathcal{L}}(f)$ holds f is clockwise oriented.

6. More about Cages

The following propositions are true:

(50) $\overline{\mathrm{LeftComp}(\mathrm{Cage}(C,i))^{\mathrm{c}}} = \mathrm{RightComp}(\mathrm{Cage}(C,i))$.

(51) If C is connected, then the Go-board of $\mathrm{Cage}(C,n) = \mathrm{Gauge}(C,n)$.

(52) If C is connected, then $\mathrm{N\text{-}min}\, C \in \mathrm{rightcell}(\mathrm{Cage}(C,n),1)$.

(53) If C is connected and $i \leqslant j$, then $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,j)) \subseteq \overline{\mathrm{RightComp}(\mathrm{Cage}(C,i))}$.

(54) If C is connected and $i \leqslant j$, then $\mathrm{LeftComp}(\mathrm{Cage}(C,i)) \subseteq \mathrm{LeftComp}(\mathrm{Cage}(C,j))$.

(55) If C is connected and $i \leqslant j$, then $\mathrm{RightComp}(\mathrm{Cage}(C,j)) \subseteq \mathrm{RightComp}(\mathrm{Cage}(C,i))$.

7. Preparing the Internal Approximation

Let us consider C, n. The functor $\mathrm{X\text{-}SpanStart}(C,n)$ yielding a natural number is defined as follows:

(Def. 2) $\mathrm{X\text{-}SpanStart}(C,n) = 2^{n -' 1} + 2$.

Next we state three propositions:

(56) $\mathrm{X\text{-}SpanStart}(C,n) = \mathrm{Center}\,\mathrm{Gauge}(C,n)$.

(57) $2 < \mathrm{X\text{-}SpanStart}(C,n)$ and $\mathrm{X\text{-}SpanStart}(C,n) < \mathrm{len}\,\mathrm{Gauge}(C,n)$.

(58) $1 \leqslant \mathrm{X\text{-}SpanStart}(C,n) -' 1$ and $\mathrm{X\text{-}SpanStart}(C,n) -' 1 < \mathrm{len}\,\mathrm{Gauge}(C,n)$.

Let us consider C, n. We say that n is sufficiently large for C if and only if:

(Def. 3) There exists j such that $j < \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $\mathrm{cell}(\mathrm{Gauge}(C,n), \mathrm{X\text{-}SpanStart}(C,n) -' 1, j) \subseteq \mathrm{BDD}\, C$.

One can prove the following propositions:

(59) If n is sufficiently large for C, then $n \geqslant 1$.

(60) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_1 be natural numbers. Suppose that

(i) $\mathrm{left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_1)$,

(iv) $\langle i_1, j_1 + 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1, j_1 + 1)$.

Then $\langle i_1 -' 1, j_1 + 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(61) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C, n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_1 be natural numbers. Suppose that

(i) $\mathrm{left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C, n))$ meets C,

(ii) $\langle i_1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C, n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C, n) \circ (i_1, j_1)$,

(iv) $\langle i_1 + 1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C, n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C, n) \circ (i_1 + 1, j_1)$.

Then $\langle i_1 + 1, j_1 + 1 \rangle \in$ the indices of $\mathrm{Gauge}(C, n)$.

(62) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C, n)$ and $\mathrm{len}\, f > 1$. Let j_1, i_2 be natural numbers. Suppose that

(i) $\mathrm{left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C, n))$ meets C,

(ii) $\langle i_2 + 1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C, n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C, n) \circ (i_2 + 1, j_1)$,

(iv) $\langle i_2, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C, n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C, n) \circ (i_2, j_1)$.

Then $\langle i_2, j_1 -' 1 \rangle \in$ the indices of $\mathrm{Gauge}(C, n)$.

(63) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C, n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_2 be natural numbers. Suppose that

(i) $\mathrm{left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C, n))$ meets C,

(ii) $\langle i_1, j_2 + 1 \rangle \in$ the indices of $\mathrm{Gauge}(C, n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C, n) \circ (i_1, j_2 + 1)$,

(iv) $\langle i_1, j_2 \rangle \in$ the indices of $\mathrm{Gauge}(C, n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C, n) \circ (i_1, j_2)$.

Then $\langle i_1 + 1, j_2 \rangle \in$ the indices of $\mathrm{Gauge}(C, n)$.

(64) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C, n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_1 be natural numbers. Suppose that

(i) $\mathrm{front\_left\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C, n))$ meets C,

(ii) $\langle i_1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C, n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C, n) \circ (i_1, j_1)$,

(iv) $\langle i_1, j_1 + 1 \rangle \in$ the indices of $\mathrm{Gauge}(C, n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C, n) \circ (i_1, j_1 + 1)$.

Then $\langle i_1, j_1 + 2 \rangle \in$ the indices of $\mathrm{Gauge}(C, n)$.

(65) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to Gauge(C, n) and len $f > 1$. Let i_1, j_1 be natural numbers. Suppose that

(i) front_left_cell$(f, \operatorname{len} f -' 1, \operatorname{Gauge}(C, n))$ meets C,
(ii) $\langle i_1, j_1 \rangle \in$ the indices of Gauge(C, n),
(iii) $f_{\operatorname{len} f -' 1} = \operatorname{Gauge}(C, n) \circ (i_1, j_1)$,
(iv) $\langle i_1 + 1, j_1 \rangle \in$ the indices of Gauge(C, n), and
(v) $f_{\operatorname{len} f} = \operatorname{Gauge}(C, n) \circ (i_1 + 1, j_1)$.

Then $\langle i_1 + 2, j_1 \rangle \in$ the indices of Gauge(C, n).

(66) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to Gauge(C, n) and len $f > 1$. Let j_1, i_2 be natural numbers. Suppose that

(i) front_left_cell$(f, \operatorname{len} f -' 1, \operatorname{Gauge}(C, n))$ meets C,
(ii) $\langle i_2 + 1, j_1 \rangle \in$ the indices of Gauge(C, n),
(iii) $f_{\operatorname{len} f -' 1} = \operatorname{Gauge}(C, n) \circ (i_2 + 1, j_1)$,
(iv) $\langle i_2, j_1 \rangle \in$ the indices of Gauge(C, n), and
(v) $f_{\operatorname{len} f} = \operatorname{Gauge}(C, n) \circ (i_2, j_1)$.

Then $\langle i_2 -' 1, j_1 \rangle \in$ the indices of Gauge(C, n).

(67) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to Gauge(C, n) and len $f > 1$. Let i_1, j_2 be natural numbers. Suppose that

(i) front_left_cell$(f, \operatorname{len} f -' 1, \operatorname{Gauge}(C, n))$ meets C,
(ii) $\langle i_1, j_2 + 1 \rangle \in$ the indices of Gauge(C, n),
(iii) $f_{\operatorname{len} f -' 1} = \operatorname{Gauge}(C, n) \circ (i_1, j_2 + 1)$,
(iv) $\langle i_1, j_2 \rangle \in$ the indices of Gauge(C, n), and
(v) $f_{\operatorname{len} f} = \operatorname{Gauge}(C, n) \circ (i_1, j_2)$.

Then $\langle i_1, j_2 -' 1 \rangle \in$ the indices of Gauge(C, n).

(68) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to Gauge(C, n) and len $f > 1$. Let i_1, j_1 be natural numbers. Suppose that

(i) front_right_cell$(f, \operatorname{len} f -' 1, \operatorname{Gauge}(C, n))$ meets C,
(ii) $\langle i_1, j_1 \rangle \in$ the indices of Gauge(C, n),
(iii) $f_{\operatorname{len} f -' 1} = \operatorname{Gauge}(C, n) \circ (i_1, j_1)$,
(iv) $\langle i_1, j_1 + 1 \rangle \in$ the indices of Gauge(C, n), and
(v) $f_{\operatorname{len} f} = \operatorname{Gauge}(C, n) \circ (i_1, j_1 + 1)$.

Then $\langle i_1 + 1, j_1 + 1 \rangle \in$ the indices of Gauge(C, n).

(69) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\rm T}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_1 be natural numbers. Suppose that

(i) $\mathrm{front\_right\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_1)$,

(iv) $\langle i_1 + 1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1 + 1, j_1)$.

Then $\langle i_1 + 1, j_1 -' 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(70) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\rm T}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let j_1, i_2 be natural numbers. Suppose that

(i) $\mathrm{front\_right\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_2 + 1, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_2 + 1, j_1)$,

(iv) $\langle i_2, j_1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_2, j_1)$.

Then $\langle i_2, j_1 + 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

(71) Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\rm T}$, given n, and f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$ and $\mathrm{len}\, f > 1$. Let i_1, j_2 be natural numbers. Suppose that

(i) $\mathrm{front\_right\_cell}(f, \mathrm{len}\, f -' 1, \mathrm{Gauge}(C,n))$ meets C,

(ii) $\langle i_1, j_2 + 1 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$,

(iii) $f_{\mathrm{len}\, f -' 1} = \mathrm{Gauge}(C,n) \circ (i_1, j_2 + 1)$,

(iv) $\langle i_1, j_2 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$, and

(v) $f_{\mathrm{len}\, f} = \mathrm{Gauge}(C,n) \circ (i_1, j_2)$.

Then $\langle i_1 -' 1, j_2 \rangle \in$ the indices of $\mathrm{Gauge}(C,n)$.

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